Microfinance and Dynamic Incentives

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Abstract

Dynamic incentives, where incentives to repay are generated by granting access to future loans, is one of the methodologies used by microfinance institutions (MFIs). In this paper, I present a model of dynamic incentives where lenders are uncertain over how much borrowers value future loans. Loan terms are determined endogenously, and loans become more favorable as the probability of default becomes lower. I show that in all equilibria but one all borrowers default - including the most patient ones. I then compare equilibria with and without double-dipping. I show that when credit-constrained borrowers have access to multiple loans equilibrium loan terms have to become more favorable to overcome increased gains from default. If borrower’s access to multiple loans is unforeseen by lenders then the default rates go up in the short-run. Finally, if lenders use group loans it generates endogenous hyperbolic discounting even though the borrowers are exponential discounters.

JEL classification: C73, D82, O12, O16

Key Words: Microfinance, unsecured credit, dynamic incentives, strategic default, double-dipping.

1 Introduction

As of December 2010, there were 3,652 microfinance institutions (MFIs) reaching more than 200 million people, most of whom were among the poorest when they took their first loan (Maes and Reed, 2012). This is remarkable given the plethora of obstacles that, for a long time, have kept formal credit institutions away from financing the poor. Adverse selection, moral hazard, lack of collaterizable assets, absence of enforcement mechanisms, and high costs should have made microfinance nothing if non-existent, or at least subsidized. As an example, during pre-Grameen times in Bangladesh, loans targeting poor households by traditional banks had repayment rates as

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low as 51.6% in 1980, down to 18.8% by 1988-89, and were heavily subsidized by the government (Khalily and Meyer, 1993).

The microfinance methodologies that are responsible for microcredit success are well-known in the literature. They are group lending (where a small group of neighbors is jointly liable for individual loans), dynamic incentives (using access to future loans as incentives to repay the current one), regular-repayment schedules and using collateral substitutes (Morduch, 1999). Among the four, group lending received the most attention, as it is an innovative and clever way to alleviate the problems of adverse selection and moral hazard. More recently, however, there has been a shift in focus, away from group lending and towards other aspects of microfinance loans. Fischer and Ghatak (2010) cite several factors responsible for this change, such as a decreased reliance on group lending by several major MFIs, as well as, a growing recognition of costs associated with joint liability (see also Banerjee, 2013, and references therein).1

The focus of this paper is dynamic incentives. The enforcement mechanism of dynamic incentives comes from a very simple, yet powerful, repeated-game argument. As long as a borrower is sufficiently patient the threat of limiting (either fully or partially), the borrower’s access to future loans serves as a punishment strong enough to deter the default. The contribution of this paper is that it demonstrates limitations of the dynamic incentives methodology despite the presence of sufficiently patient borrowers and full exclusion of defaulters.

The model is an infinitely repeated game, where a borrower faces an ex-post moral hazard, and default leads to a full exclusion of the defaulter. As in Ghosh and Ray (2001), parties cannot commit to contracts longer than one period. In a given period, loan terms are endogenously determined by the (correctly anticipated) probability of default. A lower probability of default in a given period means larger loans on more favorable terms. I assume that lenders are uncertain as to how much a borrower values future loans. I model it as uncertainty over the borrower’s discount factor, δ. However, it can be also modeled as uncertainty about the borrower’s outside option (those with a lower outside option value access to future loans more); or uncertainty over the borrower’s productivity growth (those with higher growth rate have a higher value of future loans); and, under some conditions, the ability to post-collateral (future loans become more attractive option when compared to losing a larger collateral if the current loan is not repaid). Lenders observe the borrower’s repayment history and update their beliefs about the borrower’s patience with every successful repayment.

The model has multiple equilibria. There is an efficient equilibrium where the risk of default is eventually eliminated. The borrower’s types with sufficiently high patience repay every period, and all less patient types eventually default and leave the game. The surprising finding, however,

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1Grameen eliminated explicit joint liability by 2002; Rhyne (2002) document how due to increased competition new MFIs in Bolivia, such as Caja los Andes, relied exclusively on individual loans to get a market advantage over group loan competitors, such as BancoSol or Prodem. In fact, currently BancoSol switched to individual credit technology (see http://www.mixmarket.org/mfi/bancosol, accessed January 2014) and Prodem uses the mix of individual and solidarity loans (see http://www.mixmarket.org/mfi/prodem-ffp, accessed January 2014). Nonetheless de Quidt et al. (2012) report that in their sample of 715 MFIs as many as 54% of loans are made under solidarity lending indicating that group loans are still widely used.
is that such an equilibrium is unique. All other equilibria entail default by all types, including the most patient ones. First, there is a continuum of equilibria where loan terms are unfavorable in the beginning and deteriorate even further with time. Since these future loans are not attractive, most types quickly default. This, in turn, rationalizes lenders providing unfavorable loans. The second class of inefficient equilibria allows for a temporary improvement in loan terms. Loan terms improve at first, but eventually begin to deteriorate and all types default. Now types with intermediate patience delay defaulting until they can gain access to larger loans. In equilibrium, lenders correctly anticipate this. As soon as more types find it optimal to renge on the debt, loan terms begin to worsen. This, in turn, destroys incentives to repay for the more patient types, which worsens loan terms even further. Eventually, all types will prefer to default regardless of their patience.

The existence of inefficient equilibria seems counterintuitive at first. With time, lenders are bound to learn that the borrower is sufficiently patient which should remove the risk of default. The intuition is misleading, however, since even the most patient types can find it optimal to default when future loans are expected to be small or otherwise unfavorable. Bond and Rai (2009) mention several cases where worries about MFIs’ financial solvency, for example, because of an exogenous hike in default rates, rapidly destroy everyone’s incentives to repay. While financial solvency is not an issue in my model, the key point remains valid. Dynamic incentives are not only about the borrower’s patience, but also about the expected value of future loans.

I consider several extensions of the benchmark model. First, I introduce “double-dipping” by allowing the borrower to take loans from multiple (two) lenders in a given period. Availability of the second loan has two opposite effects on repayment incentives: it increases gains from default and, if borrowers are credit-constrained, it increases gains from repayment and the value of future loans. I show that when a borrower is credit-constrained then, other things being equal, double-dipping (DD) equilibria lead to more favorable loans and a lower default rate than corresponding single-dipping (SD) equilibria. The reason is that along the DD-equilibria more favorable loans are required to offset increased gains from default and maintain repayment incentives. I also consider an off-equilibrium scenario when along the SD-equilibrium the borrower, unbeknown to lenders, gets access to multiple loans. I show that in this situation the default rates should increase above the equilibrium level, which is consistent with empirical evidence.

Another extension I consider is adding joint liability to the benchmark model. The model is based on and closely follows de Quidt et al. (2012). The novel insight that I add, is an observation that joint liability gives rise to endogenous hyperbolic discounting, even though borrowers themselves are exponential discounters. This occurs when a project of one borrower succeeds, and the project of the other borrower fails. To ensure access to future loans, the successful borrower has to repay both loans. Such a borrower, at period 0, will have a lower inter-temporal discount between the periods 0 and 1 than between periods \( t(>0) \) and \( t+1 \). At \( t=0 \), the successful borrower bears the full cost of repaying both loans but does not fully, or rather immediately, receives the benefit

\[ ^2 \text{When the borrower is not credit-constrained, the gains from default will not necessarily outweigh gains from repayment. Since a repaying borrower receives enough funding, default might become a less attractive option. In which case in equilibrium one does not need more favorable loan terms to ensure repayment.} \]
of doing so. When the borrower pays off both loans, the game moves to period 1 with probability 1, and the second borrower plays no role in it. An inter-temporal trade-off between $t(>0)$ and $t+1$ is different, however. Having the second borrower increases the probability of reaching $t+1$ conditional on reaching $t$ (as viewed from point 0). This is why from period 0 point of view, the inter-temporal discount between 0 and 1 is lower than between $t$ and $t+1$.

The paper is organized as follows: Section 2 provides a literature review. In Section 3, I describe and solve the main model. In Section 4, I consider model extensions. Sections 4.4 and 4.5 study double-dipping and joint liability loans. Section 4.1 shows impossibility of having separating loan contracts; Section 4.2 adds signaling; Section 4.3 suggests alternative sources of uncertainty and is supported by a model of generalized payoff functions developed in Section 7 (Appendix B). Most of the proofs are given in Section 6 (Appendix A).

2 Literature Review

The idea that unsecured debt, such as microcredit to the poor or sovereign debts, can be self-enforced in the case of repeated interactions was first formalized by Eaton and Gersovitz (1981) and further developed by Eaton et al. (1986), Grossman and Van Hyuck (1988), and many others. Bulow and Rogoff (1989) demonstrated some limitations of repeated interactions. If borrowers can save, then the rate of loan growth has to be higher than the interest rate, which eventually would become unsustainable. However, this result depends on the borrower’s ability to save and on the existence of an external authority that can enforce lenders’ commitment to future payments (Eaton, 1996, Kletzer and Wright, 2000). Among more recent papers, Albuquerque and Hopenhayn (2004) develop a model of optimal long-run lending contracts, when the debt repayment cannot be perfectly enforced. They apply their model to derive implications for firm growth, leverage and capital structure. The optimal contract involves the gradual growth of equity (at a rate equal to the interest rate) until the firm reaches its full potential and endogenous budget constrains are no longer binding. Differently from my paper, Albuquerque and Hopenhayn has no asymmetric information between the investor and the entrepreneur. Default can be avoided entirely by properly structuring the long-term debt contract.

Ghosh and Ray (2001) apply a dynamic incentives argument to the case of unsecured micro-credit loans where lenders do not communicate and borrowers’ repayment history is not publicly observable. There are two types of borrowers: myopic borrowers who always default, and borrowers with a positive discount factor, who do not default in equilibrium. While the history is not public, an individual lender can distinguish between new and old (returning) borrowers. If the borrower repays an initial (new) loan to a lender, the lender is willing to provide a better (old) loan to the borrower. An important difference from Ghosh and Ray (2001) is that, in my paper, the borrower’s history is observable by all lenders. It matters, as it removes the exclusive link between the old borrower and his lender. In Ghosh and Ray (2001), incentives to repay come from the fact that there is exactly one lender who has better information about the returning borrower and is willing to provide a better loan that is unavailable elsewhere. Another aspect where my paper is different
from Ghosh and Ray (2001), is that I characterize all equilibria including non-stationary ones. This allows for additional equilibrium scenarios, such as insufficiently patient borrowers repaying only to get access and then default on larger loans; or even sufficiently patient borrowers defaulting when future loans are not profitably enough.

In the literature, dynamic incentives are often considered in combination with progressive lending. As empirical research shows, microfinance contracts typically structure the loans in such a way that the starting loans are small but increase with each cycle. Robinson (2001) describes 18 loan programs in different countries and shows that 12 of them used progressive lending with amounts rising up to 200% of the initial loan. Armendáriz and Morduch (2005) show that Grameen Bank provides a continuing and increasing series of loans to its clients. More recently, Kumar (2012) conducts a survey of 424 women in Karnataka, India, showing that the sixth consecutive loan could be as large as 684% of the initial loan. Progressive lending can reinforce dynamic incentives, as a borrower who defaults on the current loan gives up the possibility of a larger loan(s) in the future. Ghosh and Ray (2001) can be seen as an example of progressive lending as new patient borrowers repay initial small loans to reveal their patience and get access to better loans in the future. Egli (2004) develops a two-period model, where it is divisibility of a project that allows for equilibria where deadbeat (“bad”) borrowers repay the first-period loans with non-zero probability (“reputational” and “pooling” equilibria).

My paper is different from the progressive lending literature, in that, in my framework the exclusive relationship between a borrower and a lender — either due to long-term contract as in Egli (2004) or due to asymmetry of lenders’ information about the borrower as in the O-phase in Ghosh and Ray (2001) — is impossible.³ Current loan terms are fully determined by the current probability of default. It has a direct impact on progressive lending, as small and expensive loans in my model invariably imply a higher probability of default. That can be only rationalized by worse (or eventually worse) loans in the future, as otherwise the default rate would have to be lower. As a consequence, the only efficient equilibrium in my model has the largest initial loan among all equilibria.

Among MFIs’ methodologies, group lending was and perhaps remains to be the most studied. Ghatak and Guinnane (1999) show that joint liability lending can effectively deal with a range of problems, including adverse selection, ex-post moral hazard, and lowering auditing costs. Besley and Coate (1995) develop a theoretical model of group lending under an ex-post moral hazard. They show that joint liability can improve repayment rates, as having other borrowers in your group can serve as an insurance for those borrowers whose projects generate insufficient return. At the same

³The role of non-exclusivity in my model is somewhat different from what is studied in Ales and Maziero (2011). Ales and Maziero (2011) study equilibrium of a finite-horizon Mirrleesian economy, where contracts between a worker and firms are non-exclusive and non-observable to firms. Firms offer menus to elicit workers’ productivity shocks as well as other contractual arrangements. In my paper, there is a non-exclusive aspect similar to that of Ales and Maziero. First, longer than one-period arrangements are assumed away and, second, when double-dipping is introduced the borrower’s ability to take multiple loans is similar to a worker signing multiple contracts in Ales and Maziero. The difference comes from the lenders’ side, as the elicitation motive does not play a major role in my framework. Lenders do not and, in fact, cannot structure loan contracts to reveal the borrower’s private information.
time, stronger incentives are needed for those borrowers who have to repay for their unsuccessful partners, as they have more to gain by defaulting. Among papers that study the combinations of dynamic incentives and joint liability de Quidt et al. (2012, 2013) are two papers that are particularly close to mine. de Quidt et al. (2012) compare borrowers’ well-being under different credit market structures, including both for-profit and non-profit lenders. de Quidt et al. (2013) compare borrowers’ incentives under individual liability (IL), explicit and implicit joint liability (EJ and IJ). The joint liability section in my paper is similar to the two papers above. My value-added is not in developing a new model or deriving new results. It is in the observation that group lending combined with dynamic incentives leads to endogenously arising hyperbolic discounting. When the analysis is done at the steady state this observation has little effect. However, for non-stationary equilibria borrower’s time inconsistency complicates the analysis beyond the scope of this paper.4

The effect of group loans in the presence of ex-ante moral hazard is studied in Chowdhury (2007). In the model, borrowers have access to two projects “good” and “bad”. Borrowers themselves are also of two types: those with social capital and those without. Comparing repeated lending with contingent lending, Chowdhury (2007) concludes that contingent lending (when only one borrower of a group gets a loan, and the second borrower receives a loan contingent on the first borrower’s repayment) generates more incentives than a simple, repeated loan framework. My paper, in addition to being an ex-post rather than ex-ante moral hazard model, does not introduce social capital. In particular, $S$-borrowers in Chowdhury (2007) repay to avoid instantaneous social punishment. In my model, the sole source of incentives is dynamic incentives.

A rapid growth in a number of microfinance institutions, as well as the entry of commercial lenders, led to a question of how increased competition affects the microcredit industry. While increased competition can lead to a decline in the interest rates5 the total effect appears to be negative. It diminishes the ability of MFIs to reach the poorest (McIntosh and Wydick, 2005), decreases information sharing among lenders (Ghosh and Ray, 2001, Luoto et al., 2007) and enables borrowers to take multiple loans, or double-dip, which weakens their repayment incentives (McIntosh and Wydick, 2005, McIntosh et al., 2005, Armendáriz and Morduch, 2005, Guha and Chowdhury, 2013). Luoto, McIntosh and Wydick (2007) document the increased awareness of the necessity of information sharing among microfinance lenders in South Asia and Latin America and their initiatives to implement credit bureaus to address the information sharing problem. Guha and Chowdhury (2013) provide a model of micro-finance competition, where MFIs’ motivation is the weighted average of profit and the borrower’s welfare. Borrowers face ex-ante moral hazard, hyperbolic discounting is commonly used to model savings’ decisions. In development context Fischer and Ghatak (2010) use hyperbolic discounting to explain a high-frequency repayment that microfinance contracts require. They argue that for present-biased borrowers, frequent repayment increases the size of incentive compatible loan, but has ambiguous welfare implications, as it can lead to over-borrowing. 

Fernando (2006) and Porteous (2006) provide evidence of declining interest rates in Bolivia and Cambodia due to increased competition. Rosenberg et al. (2013) use a large data set to provide a detailed analysis of changes to interest rates between 2004 and 2011. They show that interest rates declined through 2007 but not afterwards. Furthermore, rates for MFIs focusing on low-end borrowers and unregulated microlenders have risen while rates for banks have dropped.
and taking more than one loan is always inefficient and always leads to default. Guha and Chowdhury shows that increased competition can have opposing effects on the borrower’s well-being and interest rates. My paper differs from Guha and Chowdhury, in that it is a model of ex-post moral hazard and in that taking the second loan can be efficient, when the single-loan borrower is credit-constrained.

3 Model

3.1 Setup

Consider infinitely-repeated interactions between a risk-neutral borrower and risk-neutral lenders. The borrower has access to a project with return $F(K)$ but needs external funds to produce output. Function $F$ satisfies Inada conditions, that is, it is an increasing, concave function such that $F(0) = 0$, $F(\infty) = F'(0) = \infty$ and $F'(\infty) = 0$. The borrower discounts future payoffs with discount factor, $\delta$, which is the borrower’s private information and is unknown to lenders. Lenders have a prior that $\delta$ is distributed in $0 \leq [\delta_{\min}, \delta_{\max}] \leq 1$ with cdf $\Phi(\delta)$. I assume that $\Phi(\delta)$ is differentiable, and has a strictly positive density on $[\delta_{\min}, \delta_{\max}]$.

There is no production uncertainty, however, lenders face risk due to the borrower’s ex-post moral hazard. The borrower, once the output is produced, can choose to default on the received loan. I assume that the borrower has no collaterizable assets and that no enforcement by legal authorities is available. Instead, lenders rely on dynamic incentives as an enforcement mechanism. As long as the borrower does not default, lenders are willing to provide new loans. If the borrower defaults, however, no future loans will be given by any lender.

Time is discrete. The borrower cannot save, has no assets and, therefore, needs to borrow capital every period to finance the project. In period $t$, assuming no prior defaults, the borrower receives a loan $(K_t, R_t)$ where $K_t$ is the loan size, and $R_t$ is the interest rate. Except for Section 4.4, I assume that the borrower can take one loan only. I further assume that the borrower and the period $t$ lender cannot commit to a longer than one period contract. Upon receiving the loan, the borrower produces output $F(K_t)$ and then decides whether to repay the debt or not. In the former case, the borrower pays back $R_t K_t$ and keeps the rest, i.e. $F(K_t) - R_t K_t$, to himself. If the borrower defaults the borrower keeps the entire output $F(K_t)$ to himself. Thus, the difference between the borrower’s payoff from default and from repayment is $R_t K_t$. If the borrower repays the loan, the game continues into period $t + 1$.

In period $t$, lenders believe, correctly in equilibrium, that the probability of repayment is equal to $q_t$. I assume that loan terms, $(K_t, R_t)$, are determined by $q_t$, that is $K_t = K(q_t)$ and $R_t = R(q_t)$. Functions $K(\cdot)$ and $R(\cdot)$ are determined based on lenders’ mission and the market structure, as will be explained later. I assume that $K(0) = 0$ and

**Loan Term Monotonicity (LTM):** Loan size, $K(q)$, borrower’s gain from repayment, $F(K(q)) - R(q) K(q)$, and borrower’s gain from default, $R(q) K(q)$, are strictly increasing and continuously
differentiable functions when $q > 0$.

When the LTM is satisfied, a higher $q$ is beneficial for the borrower, since the borrower receives a larger loan and has more funds left after repayment. Furthermore, the LTM allows dynamic incentives to be combined with progressive lending. Since larger loans are more profitable, allowing future access to larger loans reinforces borrower’s incentives to repay. Notably, the LTM also assumes that the gains from default on larger loans, which are principal and interest payments, are larger. Thus, a borrower expecting larger loans in the future has a non-trivial trade-off between defaulting now and confiscating a smaller amount of wealth, versus defaulting later and confiscating a larger amount.\footnote{Banerjee (2001) reports studies indicating that bigger loans are associated with lower interest rates which is consistent with an assumption that $K(q)$ is an increasing function. Evidence supporting the assumption that $KR$ is an increasing function of $K$ can be found in Armendáriz and Morduch (2005, Table 5.3).}

To keep notations simple, I will often write the loan size and interest rate in period $t$ as $K_t$ and $R_t$, though it should be remembered that they are functions of $q_t$.

Functions $K(\cdot)$ and $R(\cdot)$ are determined based on the lenders’ mission in the case of non-profits and market structure in the case of for-profits. There are several possibilities related to those considered in the literature.

- **Non-profit borrower’s welfare maximizing MFI:** Assume that an MFI’s opportunity cost of funds is $r$. Given $q_t$, an MFI’s expected payoff net of opportunity cost should be equal to zero, that is $E_t(R_tK_t) = q_tR_tK_t + (1-q_t)0 - rK_t = 0$ so that
  \[ R_t = R(q_t) = \frac{r}{q_t}. \] (1)

The loan size then is determined, as to maximize the borrower’s payoff conditional on payback, that is $K_t = \text{arg max}_K F(K) - R_tK$. Since $R_t$ is a decreasing function of $q_t$, it follows from the envelope theorem that $F(K_t) - R_tK_t$ is an increasing function of $q_t$. Thus for any $F$, such that $R_tK_t = F'(K_t)K_t$ is strictly increasing, the LTM is satisfied. An example of such function is $F(K) = AK$.\footnote{Formally, that $KF'(K)$ is an increasing function of $K$ is equivalent to $-KF''(K) < 1$. That means that function $F$ cannot be too concave, at least on interval $[0, K(1)]$ so that its marginal productivity does not decline too quickly.}

- **For-profit competitive lender:** This case is similar to the non-profit MFI case above. The loans provided by a competitive lender should yield zero (net of opportunity cost) expected payoff. Thus $R(\cdot) = r/q_t$. The loan size is the same as in the case of non-profit MFI because perfect competition forces lenders to maximize the borrower’s welfare. If not another lender would be willing to provide $K_t$ that maximizes the borrower’s repayment payoff.

- **Non-profit outreach maximizing MFI:** In my model, all borrowers are served as long as they do not default. In particular, all borrowers receive a loan at $t = 0$. In this sense, assuming that MFIs solely maximize an outreach does not put restrictions on loan terms. We will return to outreach-maximizing MFIs later, when comparing equilibrium outcomes.
• Myopic for-profit not perfectly competitive lender: Assume that \( p(K_t) = R_t \) is the inverse demand function given \( R_t \). For instance, if the borrower’s demand for funding is such that it maximizes the borrower’s repayment payoff, then the inverse demand function is \( F'(K_t) = R_t \). For a given \( q_t \), the lender maximizes the current expected payoff \( q_t R_t K_t - r K_t = q_t \cdot p(K_t) K_t - r K_t \). Assume that the inverse demand is such that the marginal revenue is decreasing, and that the optimal loan is smaller than it would be under the perfect competition, \( F'(K_t) \geq r / q_t \). Then it is immediate to verify that both \( R_t K_t(= p(K_t) K_t) \) and \( F(K_t) - R_t K_t \) are increasing functions of \( q_t \). The LTM is satisfied.

In all of the examples above, the marginal product is equal to the interest rate. Empirical evidence, however, indicates that microloan recipients have a marginal product above the interest.\(^8\) The LTM assumption allows for such loans as well. For example, when \( R \) is constant and \( K(q) \) is a strictly increasing function such that \( F'(K(1)) \geq R \) the LTM is satisfied.

Let \( \varepsilon(q_t) \), or for brevity \( \varepsilon_t \), denote \( K_t R_t / F(K_t) \), so that \( \varepsilon_t \) is equal to the share of the output that the borrower must return to the lender. Then the repayment payoff in period \( t \) can be written as

\[
F(K_t) - R_t K_t = F(K_t)(1 - \varepsilon_t).
\]

When the loan terms are determined so that the borrower’s payoff, conditional on repayment, is maximized then \( \varepsilon_t \) is simply an elasticity of the production function at point \( K_t \). In the special case of \( F(K) = AK^\alpha \), it is constant, that is \( \varepsilon_t = \alpha \).

I assume that \( \delta_{\text{max}} > \varepsilon(q_t) \) for any \( q_t \in [0,1] \) and that \( \varepsilon(1) > \delta_{\text{min}} \). The former assumption guarantees the existence of the equilibrium where some types never default. Otherwise, there is no type patient enough for dynamic incentives to work. The latter assumption guarantees that there are also impatient types that will default. In the case of production function \( F(K_t) = AK^\alpha \) and a zero-profit lender (whether a borrower’s welfare maximizing MFI or a competitive for-profit) these assumptions are satisfied as long as \( \delta_{\text{min}} < \alpha < \delta_{\text{max}} \).

Given lenders’ beliefs \( \{q_t\} \), the borrower’s maximization problem is to determine time \( 0 \leq T \leq \infty \) to default (\( T = \infty \) means that the borrower never defaults) that maximizes

\[
\max_T \left\{ \sum_{t=0}^{T-1} (1 - \varepsilon_t) F(K_t) \delta^t + F(K_T) \delta^T \right\}, \tag{2}
\]

The optimal default time depends on \( \delta \), and I will denote the solution to (2) as \( T(\delta) \).

In the equilibrium of the model the following conditions should hold. The borrower correctly anticipates lenders’ beliefs \( \{q_t\} \) which determine the sequence of loan terms \( \{K_t, R_t\} \). Given \( \{K_t, R_t\} \), the borrower chooses the optimal time, \( T(\delta) \), to default. Lenders, given borrower’s strategy \( T(\delta) \) and history of the game, correctly estimate the probabilities of payback \( \{q_t\} \), which determine loan terms \( \{(K_t, R_t)\} \).

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\(^8\)See e.g. de Mel et al. (2008) and McKenzie and Woodruff (2008). Karlan et al. (2012) report somewhat mixed evidence. Nonetheless, their interpretation is that their own interpretation is that firms in their sample are credit constrained.
Definition 3.1. For a given $K(q)$ and $R(q)$ the equilibrium of the model is a sequence of beliefs $\{q_t\}_{t=0}^{\infty}$ and borrower’s strategy $T(\delta)$ such that

i) Given $\{q_t\}$, the borrower’s optimal default time is given by $T(\delta)$, which is a solution to (2);

ii) Given borrower’s strategy $T(\delta)$ and prior $\Phi(\delta)$ for each period $t$ lenders correctly estimate the probability of payback $q_t$, which determines loan terms $(K_t, R_t)$ in period $t$.

3.2 Complete Information Case

As a useful benchmark, I first solve the model where $\delta$ is common knowledge. Equilibrium in the model with complete information is defined similarly to Definition 3.1. A borrower plays his best response given lenders' beliefs $\{q_t\}$. Lenders’ beliefs are correct, given the borrower’s strategy. I will restrict my attention to pure-strategy equilibria, as they are more relevant to the incomplete-information framework where a measure of indifferent types is equal to zero. The probability of default in a given period then is either 0 or 1. Only two scenarios can occur in equilibrium: a riskless scenario where $q_t = 1$ for every period and no default; and a no-loan scenario.

First, consider sequence $q_t \equiv 1$. Since $\{q_t\}$ is constant, the borrower will either default immediately or will not default at all. The latter is optimal when

$$F(K(1)) \leq \sum_{t=0}^{\infty} \delta^t (1 - \varepsilon(1)) \cdot F(K(1)) = (1 - \varepsilon(1)) \cdot F(K(1)) \frac{1}{1 - \delta} \iff \delta \geq \varepsilon(1).$$

Thus when $\delta \geq \varepsilon(1)$, beliefs $q_t \equiv 1$ are rational. Naturally, in order for dynamic incentives to work, the borrower should be sufficiently patient. The no-loan beliefs $q_t \equiv 0$ is also an equilibrium since the borrower would default on any loan received in period $t$, regardless of $\delta$, and, therefore, expectations $q_t \equiv 0$ are rational.\footnote{No other equilibrium in pure strategies exists. Since even the most favorable stream of loans is not sufficient to deter the default if a borrower’s $\delta < \varepsilon(1)$, equilibrium with positive loans can exist only if $\delta \geq \varepsilon(1)$. Sequence $q_0 = \cdots = q_t = 1$ and $q_s = 0$ when $s > t$ is not an equilibrium. The borrower will default at $t$ making $q_t = 1$ an incorrect belief. Thus, there exists $t$ such that $q_t = 0$ and $q_{t+1} = 1$. This is not an equilibrium either. A lender in period $t$ could offer a riskless loan based on $q_t = 1$, which would be repaid by the borrower. This is because it is strictly more profitable to default at $t+1$ than at $t$ if $\delta > \varepsilon(1)$, and is just as profitable if $\delta = \varepsilon(1)$. This means $q_t = 0$ is an incorrect belief. Thus, no other pure-strategy equilibrium exists.}

In what follows, I will refer to types with $\delta \geq \varepsilon(1)$ as patient and types with $\delta < \varepsilon(1)$ as impatient. The reasoning above shows that in the complete information case only patient types are patient enough for dynamic incentives to work. No loans are possible in equilibrium if the borrower is impatient.

3.3 Incomplete Information Case. Derivation of Equilibrium Dynamics

We first solve for the borrower’s optimal time to default given lenders’ beliefs $\{q_t\}$. Let $K_t = K(q_t)$ and $K_{t+1} = K(q_{t+1})$ be loan levels that correspond to $q_t$ and $q_{t+1}$. Let $\delta_t$ be a discount factor such that the borrower with $\delta_t$ is indifferent between defaulting at moment $t$ and moment $t + 1$:

$$F(K_t) = F(K_t)(1 - \varepsilon(q_t)) + \delta_t F(K_{t+1}), \quad (3)$$
which is equivalent to
\[ \delta_t = \varepsilon(q_t) \frac{F(K_t)}{F(K_{t+1})}. \tag{4} \]

Equation (3) can be viewed as an analogue of the Euler equation. The Euler equation states that the agent is indifferent to reallocating an infinitesimal amount of consumption between periods \( t \) and \( t+1 \). In my model, the choice variable in period \( t \) is binary. Equation (3) states that the agent with \( \delta = \delta_t \) is indifferent between default at \( t \) and \( t+1 \).

From (3), all types with \( \delta < \delta_t \) prefer default at \( t \) to default at \( t+1 \); all types with \( \delta > \delta_t \) will prefer default at \( t+1 \) to default at \( t \).\(^{10}\) Proposition 3.2 below shows that in equilibrium, \( \{\delta_t\} \) has to be a strictly increasing sequence and that for a borrower with \( \delta_t < \delta < \delta_{t+1} \) it is optimal to default at \( t+1 \).

**Proposition 3.2.** In any equilibrium \( \{\delta_t\} \) is a strictly increasing sequence. The borrower’s optimal strategy \( T(\delta) \) is then defined as follows: for types with \( \delta \in (\delta_t, \delta_{t+1}) \) it is optimal to default at period \( t+1 \).

The intuition is straightforward. Let \( \pi_t(\delta) \) denote the utility of the borrower with discount factor \( \delta \) from default at \( t \). If \( \delta \)-sequence is strictly increasing then
\[ \pi_0(\delta) < \pi_1(\delta) \cdots < \pi_t(\delta) < \pi_{t+1}(\delta) > \pi_{t+2}(\delta) > \ldots. \tag{5} \]

Indeed, since \( \{\delta_t\} \) is increasing, \( \delta > \delta_t \) implies \( \delta > \delta_{\tau} \) for any \( \tau \leq t \). Therefore, \( \pi_{\tau}(\delta) < \pi_{\tau+1}(\delta) \) for any \( \tau \leq t \). Similarly, \( \delta < \delta_{t+1} \) implies \( \delta < \delta_{\tau} \) for any \( \tau \geq t+1 \) and, therefore, \( \pi_{\tau}(\delta) > \pi_{\tau+1}(\delta) \) for any \( \tau \geq t+1 \).

The reason why \( \delta \)-sequence has to be increasing is as follows. The value of \( \delta_t \) reflects the intertemporal comparison between loans in periods \( t \) and \( t+1 \). Higher (lower) values of \( \delta_t \) mean that the loan in period \( t \) is more (less) favorable than the loan in period \( t+1 \). When \( \delta_t > \delta_{t+1} \) it means, roughly speaking, that loans in periods \( t \) and \( t+2 \) are better than the loan at \( t+1 \). But then no one will default at \( t+1 \). Those who were patient enough to wait until \( t+1 \), will prefer to wait until a better loan at \( t+2 \). This leads to a contradiction. If no one defaults at \( t+1 \), then \( q_{t+1} = 1 \) and the loan at \( t+1 \) is actually the most favorable one.

Proposition 3.2 determined a borrower’s optimal strategy, \( T(\delta) \), or equivalently \( \{\delta_t\} \) given lenders’ beliefs \( \{q_t\} \). Next we derive lenders’ beliefs given the borrower’s strategy. In the beginning of period \( t \), it is known that \( \delta > \delta_{t-1} \) and that borrowers with \( \delta \in (\delta_{t-1}, \delta_t) \) will default. Therefore, the probability of payback is equal to the probability of \( \delta > \delta_t \) conditional on \( \delta > \delta_{t-1} \):
\[ q_t = Prob\{\delta \geq \delta_t|\delta \geq \delta_{t-1}\} = \frac{1 - \Phi(\delta_t)}{1 - \Phi(\delta_{t-1})}, \]
where \( \delta_{t-1} \) is defined as \( \delta_{\text{min}} \).

\(^{10}\)Note that (3) does not mean that for \( \delta < \delta_t \) (\( \delta > \delta_t \)) it is optimal to default at \( t \) (at \( t+1 \)). It does, however, mean that types with \( \delta < \delta_t \) (\( \delta > \delta_t \)) will not default at \( t+1 \) (at \( t \)), since default at \( t \) (at \( t+1 \)) gives a strictly higher profit.
Combining the two, we have that the equilibrium dynamic should satisfy system:

\[
\begin{align*}
\delta_t &= \varepsilon(q_t) \frac{F(K_t)}{F(K_{t+1})}, \quad t \geq 0 \\
q_{t+1} &= \frac{1 - \Phi(\delta_{t+1})}{1 - \Phi(\delta_t)}, \quad t \geq -1
\end{align*}
\]  

Given \( q_0 \), one can unravel the equilibrium dynamic as follows.\(^{11}\) From the second equation, which when \( t = 0 \) becomes \( q_0 = 1 - \Phi(\delta_0) \), we determine \( \delta_0 \). Using the first equation, \( q_0 \) and \( \delta_0 \) determine \( q_1 \). Given \( q_1 \), one can use the second equation of (6) to determine \( \delta_1 \) and so on. Having \( K(q) > 0 \) ensures that \( q_{t+1} > 0 \), however, it is possible to get \( q_{t+1} > 1 \) when solving (6).\(^{12}\) It happens if \( q_t \) is so high that even most favorable (riskless) loan in \( t + 1 \) is not good enough to guarantee the repayment rate of \( q_t \). For \( \delta \)-sequence, as long as \( 0 \leq q_t \leq 1 \) for all \( t \), the second equation of (6) ensures that \( \delta_t \in [\delta_{t-1}, \delta_{max}] \). Thus, a solution to (6) is an equilibrium iff \( q_t \leq 1 \) for every \( t \). In Proposition 3.4, I will prove that equilibrium exists.

The next proposition shows an equilibrium with higher initial beliefs has better loan terms and lower default rates.\(^{13}\)

**Proposition 3.3.** Let \( \{(q_t, \delta_t)\} \) be an equilibrium with initial conditions \( (q_0, \delta_0) \) and \( \{(Q_t, \Delta_t)\} \) be an equilibrium with initial conditions \( (Q_0, \Delta_0) \). If \( q_0 > Q_0 \) then \( q_t > Q_t \) and \( \delta_t < \Delta_t \) for every \( t \).

To understand why Proposition 3.3 holds note that \( q_t > Q_t \) means that the \( q \)-loan is more favorable and, therefore, more tempting to default upon. In order for \( q_t \) to be rational, the \( t + 1 \)-loan has to be more favorable to offset larger gains from default. Thus, \( q_{t+1} > Q_{t+1} \) and the default rate at \( t + 1 \) must be lower. This, in turn, implies \( \delta_{t+1} < \Delta_{t+1} \). Similarly, lower \( \delta_t \) means that \( t + 1 \)-loan must be more favorable because more impatient borrowers are willing to wait until \( t + 1 \). That again means that \( q_{t+1} > Q_{t+1} \) and then \( \delta_{t+1} < \Delta_{t+1} \).

System (6) has two limit points, \((0, \delta_{max})\) and \((1, \varepsilon(1))\), and they correspond to the two equilibria of the complete information model. The former is the no-loan equilibrium, and the latter is an efficient equilibrium where patient types receive riskless loans and never default. Proposition 3.4 shows that equilibria exist and, as expected, any equilibrium trajectory must converge to one of these two limit points. Thus, the two equilibria of the complete information case are the only possible long-run outcomes.

**Proposition 3.4.** Equilibrium trajectories exist and converge to either the efficient steady state \((1, \varepsilon(1))\) or the no-loan steady state \((0, \delta_{max})\).

\(^{11}\)It is worth mentioning that even though the value of \( q_0 \) pinpoints the equilibrium dynamic it is not an exogenous parameter. It represents lenders’ beliefs in period 0, which are determined in equilibrium by the rational expectation condition, that is beliefs have to be correct given the borrower’s strategy. As a side remark, note that \( \delta_0 \) also pinpoints the equilibrium dynamic in which case one could compare equilibria in terms of higher or lower initial default rate.

\(^{12}\)For example, consider \( q_0 = 1 \). Then from the second equation of (6) \( \delta_0 = \delta_{min} \), that is nobody defaults at \( t = 0 \). Since we assumed that \( \varepsilon(1) > \delta_{min} \), it would imply that \( K_1 > K_0 = K(1) \) and therefore \( q_1 \) has to be greater than 1.

\(^{13}\)Note that in my model an equilibrium is an infinite sequence that specifies lenders’ beliefs and borrower’s decision for every period \( \{(q_t, \delta_t)\} \). Thus, multiplicity of equilibria means that there exist many equilibrium sequences. However, within a given equilibrium in a given period there is no multiplicity. That is, if equilibrium is \( \{(q^*_t, \delta^*_t)\} \) then the only thing that can happen at period \( t \) is that types with \( \delta^*_{t-1} < \delta \leq \delta^*_t \) default and lenders’ beliefs are \( q^*_t \).
The next proposition is one of the main results in the paper, and it highlights the limitations of dynamic incentives. It shows that, while there exists an equilibrium that converges to \((1, \varepsilon(1))\), such an equilibrium is unique. The result seems to be counter-intuitive at first. Proposition 3.2 established that in any equilibrium the \(\delta\)-sequence must be increasing and, therefore, eventually lenders learn that the borrower is sufficiently patient. Dealing with a patient borrower should imply that there is a smaller risk of default and lenders should be willing to provide more favorable loans which would give incentives to repay them.

The reason why this intuition fails, is that the knowledge of the borrower’s high patience does not necessarily remove the risk of default. If the future loans are not favorable, then even the most patient types will prefer to confiscate the debt which, in turn, would justify a low confidence and unfavorable loan terms, thereby justifying defaults by patient types in the first place. This logic is similar to Band and Rai (2009) who documented several cases where concerns about MFI’s financial solvency quickly destroyed incentives to repay. In my model, there is no question about MFI’s solvency. However, the result has the same spirit. When future loans are less valuable, it undermines the power of dynamic incentives.

**Proposition 3.5.** There exists an equilibrium that converges to \((1, \varepsilon(1))\). This equilibrium is unique.

To understand the reason why the efficient equilibrium is unique consider two equilibria \((q_t, \delta_t)\) and \((q'_t, \delta'_t)\) that converge to \((1, \varepsilon(1))\) and assume that \(q_0 > q'_0\). From Proposition 3.3, it follows that \(q_t > q'_t\) and \(\delta_t < \delta'_t\) for all \(t\). Since \(\{\delta_t\}\) and \(\{\delta'_t\}\) converge to \(\varepsilon(1)\) and \(\delta_t < \delta'_t \leq \varepsilon(1)\), the mass of types that will default after period \(t\) is always larger along the first equilibrium. It has to be the case then, that at some point \(t\) the probability of default along the second equilibrium is less than or equal to the probability of default along the first one \((q_t \leq q'_t)\). This would contradict Proposition 3.3 and, therefore, \((q'_t, \delta'_t)\) cannot converge to \((1, \varepsilon(1))\).

### 3.4 Types of Equilibria

To describe properties of different equilibrium dynamics the following two observations will be helpful. First, there is a multiplicity of equilibria and the equilibrium converging to \((1, \varepsilon(1))\) has the highest \(q_0\) among all equilibria. Indeed, let \(q_0\) be the initial belief that corresponds to the efficient equilibrium. There is no equilibrium with \(q_0 > q'_0\). Otherwise, by Proposition 3.3, the \(q\)-sequence of such equilibrium would have to converge to 1, which would contradict the uniqueness result of Proposition 3.5. From Proposition 3.3 also follows that any initial belief, such that \(q_0 < q'_0\), will result in an equilibrium trajectory, as \(q_t < q'_t \leq 1\) for any \(t\).

Second, there are only three possibilities for a loan-term dynamic. Loan terms can improve with time; loan terms can deteriorate with time; or loan terms improve at first, but then deteriorate. This is because if \(q_t > q_{t+1}\), then \(q_{t+1} > q_{t+2}\), and, therefore \(q_{t+s} > q_{t+s+1}\) for any \(s > 0\). Indeed, if \(q_t > q_{t+1}\) then from (6)

\[
F(K_{t+2}) = \frac{\varepsilon_{t+1}F(K_{t+1})}{\delta_{t+1}} < \frac{\varepsilon_tF(K_t)}{\delta_{t+1}} < \frac{\varepsilon_tF(K_t)}{\delta_t} = F(K_{t+1}).
\]
The first inequality follows from the LTM, and the second from Proposition 3.2. Intuitively, if loan terms begin to deteriorate, it damages the borrower’s incentive to repay which in turn justifies a further loan deterioration.

The example of the efficient equilibrium dynamic can be seen on Figure 1. Figure 1, as well as Figures 2 and 3 later, is plotted for $\delta \sim U[0,1]$, production function $F(K) = \sqrt{K}$, and under the assumption that the amount of capital is determined in order to maximize $F(K_t) - R_t K_t$. The dashed line, which plots the $\delta$-sequence, converges to $\varepsilon(1)(=1/2)$ meaning that all types with $\delta < \varepsilon(1)$ sooner or later default, and types with $\delta \geq \varepsilon(1)$ stay in the game permanently. The $q$-sequence, which is a solid line, converges to 1 as the risk of default eventually disappears.

The efficient equilibrium is an example of equilibrium with progressive lending, where the initial loan terms are small and, as long as the borrower repays, loan terms improve until the risk of default is fully eliminated. Differently from earlier papers, however, progressive lending only works in the efficient equilibrium and it cannot start with loans that are too small. In fact, $\bar{q}_0$ is the highest among all equilibria. The difference is due to the fact that, in my model, any exclusive borrower-lender relationship, such as long-term contracts, is impossible. Loan terms in period $t$ are determined solely based on a likelihood of default in period $t$. As shown in Proposition 3.3, a lower initial loan must imply higher default rates which are, in turn, rationalized by even worse future loans.

![Figure 1: Pareto-efficient equilibrium. The solid line is $q$-sequence, that is lenders’ confidence. It converges to 1 and the risk of default is eventually eliminated. The dashed line is the $\delta$-sequence. At the beginning of moment $t+1$ types in $[0, \delta_t]$ have already defaulted and types in $[\delta_t, 1]$ are still in the game. The $\delta$-sequence converges to 1/2 which is $\varepsilon(1)$ in this example. The figure is plotted assuming that $\delta \sim U[0,1]$ and $F(K) = \sqrt{K}$.](image)

As argued above, among inefficient equilibria there are two types depending on whether lenders’

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14 In addition to the papers cited in the literature review, Thomas and Worrall (1994) study how to structure a contract between a host country and a multinational corporation that is interested in investing in the country given that any such contract has to be self-enforced. As with progressive lending, investment is initially underprovided but increases over time and, depending on parameters, can reach the efficient level. Watson (1999) and (2002) develop a model of a dynamic partnership game with two-sided incomplete information. Watson (1999) and (2002) show that it is possible for players to eventually achieve a cooperative equilibrium by starting small, no matter how pessimistic they are at the beginning of the partnership.
confidence falls from the beginning \((q_1 \leq q_0)\), or whether it increases at first \((q_1 > q_0)\).\(^{15}\) An example of the equilibrium with no loan term improvement can be seen in Figure 2. Along these equilibria repayment rates, \(q_t\), are quickly decreasing. The value of future loans is low to begin with and becomes even lower with every new period. Eventually, even the most patient types default. Some evidence from studies comparing the repayment rates of new borrowers with those of returning borrowers suggest that the repayment rates among the latter tend to be lower (Vogelgesang, 2003, Pollio and Obuobie, 2010), which corresponds to equilibria with declining \(\{q_t\}\).

![Figure 2: The solid line is \(q\)-sequence, that is lenders’ confidence. The dashed line is \(\delta\)-sequence. At the beginning of moment \(t + 1\) types in \([0, \delta_t]\) have already defaulted and types in \([\delta_t, 1]\) are still in the game. The figure is plotted assuming that \(\delta \sim U[0, 1]\) and \(F(K) = \sqrt{K}\).](image)

The example of the equilibrium dynamic with non-monotone loan terms can be seen in Figure 3. Along this equilibrium, the stream of loans is initially attractive. All but the most impatient types prefer to pay back and get access to more favorable loans. However, the funds inflow is not large enough to make these incentives permanent. Types with medium patience are paying back simply to get access to more favorable loans, which would be more profitable to default upon. Lenders correctly anticipate this, so as soon as more types find it optimal to default, the confidence level drops down, which destroys incentives for more patient types and eventually all types default.

We summarize the observations above with the following statement.

**Theorem 3.6.** There are three types of equilibria in the model:

a) (No loan term improvement) The confidence level \(\{q_t\}\) and, consequently, the loan sizes \(\{K_t\}\) decrease from the very beginning. All types eventually default.

b) (Temporary loan term improvement) Initially, the confidence level \(\{q_t\}\) and the loan sizes \(\{K_t\}\) grow. However, eventually they become decreasing sequences, and all types eventually default.

c) (Efficient equilibrium) There exists a unique value of initial confidence \(\bar{q}_0 \leq 1\), such that the corresponding equilibrium is asymptotically efficient, that is \(\bar{q}_t \to 1\), \(\bar{\delta}_t \to \varepsilon(1)\).

\(^{15}\)Both types always exist. An example of the equilibrium with \(q_1 < q_0\) is an equilibrium with \(q_0^{\text{low}}\) that is sufficiently close to 0 so that corresponding \(\delta_0^{\text{low}}\) is sufficiently close to \(\delta_{\text{max}}\). By assumption \(\delta_{\text{max}} > \varepsilon(q)\) for any \(q\). Therefore, \(\delta_0^{\text{low}} > \varepsilon(q_0^{\text{low}})\) and from the first equation of (6) then it follows that \(q_1^{\text{low}} < q_0^{\text{low}}\). Equilibria with a temporary increase of \(q\) also exist. Along the efficient equilibrium \(\bar{q}_1 > \bar{q}_0\). By continuity, this inequality will also hold for \(q_0\) sufficiently close to \(\bar{q}_0\).
Next, I compare the equilibria from the borrower’s and lenders’ point of view. First of all, regardless of $\delta$, the borrower prefers equilibria with higher $q_0$. Indeed, consider the two equilibria \{(q_t, \delta_t)\} and \{(Q_t, \Delta_t)\}, where $q_0 > Q_0$. Let $\pi_t(\delta)$ denote the borrower’s profit from default at $t$ along the $q$-equilibrium, and $\Pi_t(\delta)$ profit along the $Q$-equilibrium. Let $t^{*}$ denote the optimal time to default along the $q$-equilibrium and $T^{*}$ denote the optimal time to default along the $Q$-equilibrium. From Proposition 3.3 follows that $\pi_t(\delta) > \Pi_t(\delta)$ for every $t$. Thus, $\pi_{t^{*}}(\delta) \geq \pi_{T^{*}}(\delta) > \Pi_{T^{*}}(\delta)$ and the borrower is better off in the $q$-equilibrium.

For lenders the result is similar, though the justification depends on the mission and market structure. By the logic above, the non-profit MFI maximizing borrower’s welfare prefers equilibria with higher $q$. Competitive lenders are indifferent across equilibria as their expected profit is zero. A for-profit non-perfectly competitive lender prefers equilibria with a higher $q$ (follows from the envelope theorem). Finally, since borrowers default later in equilibria with higher $q_0$, an outreach maximizing non-profit MFIs also prefers equilibria with higher $q_0$.

I conclude this section with a quick remark on the equilibrium selection. Since, except for the case of competitive lenders, both sides strictly prefer the efficient equilibrium, i.e. the one with the highest $q_0$, and the competitive lenders are indifferent, one might expect the efficient equilibrium to become the focal equilibrium that players could coordinate on. However, first of all, it does not invalidate the message of this section, which is that dynamic incentives can go wrong even with progressive lending and patient borrowers. Second, as argued in Harsanyi and Selten (1988), in coordination games with multiple Pareto-raned equilibria, it is risk-dominant and not payoff-dominant equilibrium that is likely to be selected. In my model, the efficient equilibrium is also the riskiest as it requires the largest loans. If a lender mis-coordinates and provides a larger than an equilibrium loan, more borrowers will prefer to default on that loan. Thus, if lenders are worried about mis-coordinating, then it is an inefficient equilibrium that might get selected.
4 Extensions of the Model

4.1 Double Dipping

One of the consequences of a rapid expansion of micro-credit is that borrowers have access to multiple lenders. Guha and Chowdhury (2013) cite the Wall Street Journal’s article from November 27, 2011: “Surveys have estimated that 23% to 43% of families borrowing from micro-lenders in Tangail borrow from more than one.” This phenomenon is called double-dipping and, in this section, I study how double-dipping affects dynamic incentives.

Assume that there are two lenders who are willing to lend to the borrower and cannot prevent a given borrower from signing a loan contract with the other lenders. The loan terms provided by each lender are the same and determined by a probability of default. Each period, the borrower with no prior defaults has an option of borrowing money from one lender or two lenders. Upon getting the loan(s), the borrower use them to produce output and then decides whether to default or not. As in the main model, defaults are assumed to be publicly observable, despite a borrower taking multiple loans, and will prevent the borrower from future loans. The assumption is consistent with McIntosh et al. (2005)’s observation that “Particularly in rural areas, clients are extremely unlikely to be able to default in one joint-liability network without acquiring a reputation that would preclude their joining another.”

To make single-dipping (SD), and double-dipping (DD) frameworks, comparable I assume that, for a given $q$, single loan terms are the same in SD and DD frameworks, $(K(q), R(q))$. To what extent this assumption is realistic depends on the MFI’s mission and market structure. For instance, whether DD is available or not, a welfare-maximizing non-profit MFI will provide the loan at the level so that the marginal productivity of a single loan is equal to the loan’s interest rate. In this case, the repaying borrower will take a single loan and the effect of double-dipping will manifest itself in how the probability of default is determined. On the other hand, in the case of a for-profit lender with some market power, the assumption is less appropriate as an increased competition would affect the loan terms.

As in the SD case, the borrower’s trade-off is determined by the comparison of his gains from default and his gains from repayment. The borrower planning to default in period $t$, will take loans from both lenders at the maximum amount, $2K(q_t)$. The borrower planning to repay in period $t$, will borrow at level $K_t^*$ that maximizes his payoff after the repayment, $\max_{K \in [K(q_t), 2K(q_t)]} F(K) - R(q_t)K$. The borrower’s gain from default, therefore, is $F(2K(q_t)) - (F(K_t^*) - R(q_t)K_t^*)$.

Proposition 4.1. Let $K(q)$ and $R(q)$ satisfy the LTM. Assume also that $F'(K)K$ is an increasing function of $K$ and $R(q)$ a decreasing function of $q$. Then $\max_{K \in [K(q), 2K(q)]} F(K) - R(q)K$ and $F(2K(q)) - \max_{K \in [K(q), 2K(q)]} F(K) - R(q)K$ are increasing functions of $q$.

16 An implicit assumption in Section 3 was that the borrower always takes the full amount of loan, i. e. $F'(K(q)) \geq R(q)$. In the case of double-dipping I maintain the assumption that for a repaying borrower it is optimal to take at least one full loan. With the second loan I allow for two possibilities. Either the borrower remains credit-constrained with two loans (then $F'(2K(q)) \geq R(q)$ and $K^* = 2K(q)$), or the second loan removes credit constraints ($K^* < 2K(q)$ and $F'(K^*) = R(q)$).
Given that both repayment payoff and gains from default are increasing functions of $q$, all the results from Section 3 can be immediately extended to the case of double-dipping. For the DD-borrower to be indifferent between default in periods $t$ and $t + 1$ his $\delta$ should satisfy

$$
\delta_t = \frac{F(2K_t) - (F(K_t^*) - R_tK_t^*)}{F(2K_{t+1})}.
$$

The Bayesian update condition is the same as before,

$$
q_{t+1} = \frac{1 - \Phi(\delta_{t+1})}{1 - \Phi(\delta_t)}.
$$

These two equations determine the double-dipping equilibrium dynamic.

**Proposition 4.2.** Consider two models, one with double-dipping (DD) and one without (SD), that are based on the same $K(q)$ and $R(q)$. Then

i) The efficient steady state in equilibrium with double-dipping requires higher patience, i.e. $\delta_{dd, eff} > \delta_{sd, eff}$.

ii) Assume that loan terms are such that a double-dipping borrower is credit-constrained, i.e. $F'(2K(q)) \geq R(q)$ for every $q$. If equilibrium trajectories under SD and DD begin with the same initial $q_0$, then $q_{t+1}^{dd} > q_{t+1}^{sd}$ and $\delta_{t+1}^{dd} < \delta_{t+1}^{sd}$ for every $t > 0$.

iii) Assume a double-dipping borrower is not necessarily credit-constrained. Assume also that $F'(K)K$ is an increasing function of $K$ while $F'(K)K/F(K)$ is a weakly decreasing function of $K$. Then if equilibrium trajectories under SD and DD begin with the same initial $q_0$ then $q_{t+1}^{dd} > q_{t+1}^{sd}$ and $\delta_{t+1}^{dd} < \delta_{t+1}^{sd}$ for every $t > 0$.

**Corollary 4.3.** If Proposition 4.4 is satisfied, then $q_{0, eff}^{sd} > q_{0, eff}^{dd}$, that is efficient equilibrium along DD requires lower initial loans, i.e. starting smaller.

Double-dipping has two opposite effects on borrower’s incentives. First, the borrower’s gains from defaults are higher. Second, the borrower’s gains from repayment are higher (strictly higher whenever $K^* > K(q)$, i.e. whenever the borrower is credit-constrained under SD). Part i) of Proposition 4.4 states that along the steady state the first effect always dominates. That is higher patience is required in the presence of double-dipping.

Parts ii) and iii) compare the single-loan terms between DD and SD equilibria. It turns out the comparison is not trivial and depends on whether a double-dipping borrower remains to be credit-constrained or not. Intuitively, for the same loan terms in period $t$, the credit-constrained borrower’s gain from default doubles under DD, $F(2K_t) - (F(2K_t) - 2R_tK_t) = 2R_tK_t$, as compared to SD. The output from the future loan, however, less than doubles $F(2K_{t+1}) < 2F(K_{t+1})$ due to concavity of $F$. Therefore, to maintain repaying incentives, loan terms for future single loans under DD should become more favorable. This, in turn, implies that the equilibrium default rate should be lower. The logic is similar in spirit to de Quidt et al. (2012), where borrowers are better off under the monopolistic lender. The monopolistic lender in de Quidt et al. (2012) has to give more rent to borrowers, as their incentive constraints are tightened. Similarly, in my model, DD-loan terms should become more favorable in order to offset a higher incentive to default.
If the borrower is not credit-constrained then the borrower’s gain from default becomes a concave function of $K_t$: $F(2K_t) - (F(K^*) - R_tK^*)$. This makes a comparison between SD and DD ambiguous. The sufficient condition that guarantees that DD loans remain higher is that $K'F'(K)/F(K)$, which is elasticity of $F$, is a weakly decreasing function of $K$. To interpret this condition, note that a weakly decreasing elasticity implies $(\ln F(2K))'_K \leq (\ln F(K))'_K$. Thus, an increase in single-loan terms, i.e. $K$, leads to a higher percentage in output’s increase under SD than under DD. Therefore, to maintain the incentives along the DD-equilibrium, DD single loan terms should be more favorable.

Empirical evidence on double-dipping suggests that double-dipping can be an unanticipated shock for lenders. For instance, McIntosh et al. (2005) report a decline in repayment performance, which indicates that lenders got caught by surprise and have not adjusted to a presence of double-dipping borrowers, at least, not at the moment of the study. I conclude this section with a brief analysis of an off-equilibrium outcome along an SD-equilibrium trajectory, where at period $t$ the borrower takes two loans from two lenders unbeknown to both lenders.

Assume that $K_t$ and $K_{t+1}$ are defined according to the SD-equilibrium. Defaults in period $t$ come from types whose $\delta$ is between $\delta_{t-1}$ and $\delta_t$, where $\delta_t$ satisfies the indifference condition
$$F(K_{sd}^t) - R_tK_{sd}^t + \delta_{sd}^t F(K_{sd}^t+1) = F(K_{sd}^t).$$

With the ability to double-dip and assuming that borrower’s beliefs about future loan terms do not change, the indifference condition becomes
$$F(K^*_t) - R_tK^*_t + \delta_{dd}^t F(2K_{sd}^t+1) = F(2K_{sd}^t).$$

As before, from concavity of $F(\cdot)$ follows that $\delta_{dd}^t > \delta_{sd}^t$ (the exact proof is identical to that on p. 33 in the Appendix, where I show that the DD-efficient steady state requires higher $\delta$). Thus, during the off-equilibrium period, the default rate will go up. Whether the average loan size per loan declines depends on whether $K^*_t$ is equal to $2K_{sd}^t$ or is less. In McIntosh et al. (2005)’s study of a Uganda’s MFI, the loan size did not respond to the availability of double-dipping (p.997), which is consistent with the case of borrowers being credit-constrained, so that even those willing to repay find it optimal to take two maximal loans.17

### 4.2 Joint Liability and Dynamic Incentives

In this section, I study how joint liability affects dynamic incentives. Joint Liability (JL) is modeled similarly to de Quidt et al. (2012), except that I do not introduce social capital (Besley and Coate, 1995) into my model to keep the paper focused on dynamic incentives. This section is narrower in its scope than the rest of the paper. First, I assume that $K(q)$ and $R(q)$ are determined so that the

17I do not analyze what happens after period $t$. Generally speaking, at period $t + 1$ only types with $\delta \in [\delta_{dd}^t, \delta_{max}]$ are left in the game. But because the equilibrium logic was broken in period $t$, the value of $K_{t+1}$ (or equivalently $q_{t+1}$) is no longer restricted. Thus the $t + 1$-period situation is identical to the situation when $t = 0$, but the initial distribution of types is $\Phi(\delta|\delta \geq \delta_{dd}^t)$. In particular, there is a multiplicity of equilibria with exactly one being efficient. The efficient steady state is the one that corresponds to the DD-case.
lenders expected profit is zero and the loan amount maximizes the borrower’s repayment payoff. This is consistent with a non-profit MFI maximizing borrower’s welfare or a for-profit competitive MFI. Second, joint and individual liability will be compared only in terms of their efficient steady states. The main insight that this section adds to the literature is that joint liability gives rise to an endogenous hyperbolic discounting, despite the fact that borrowers themselves are exponential discounters.

**Individual Liability (IL):** Consider the benchmark model in Section 3, and assume that the project’s outcome is not deterministic. With probability $s \leq 1$, the project succeeds and generates return $F(K)$. With probability $1 - s$, the project fails and generates return 0. The borrower is individually liable for the loan. If the project fails the borrower cannot repay the loan and has to default. If the project succeeds, the borrower can afford to repay the loan, but may also choose a strategic default instead. Whether the default is strategic or not, the borrower loses the access to the credit market after the default.

If $1 - q$ is the probability of strategic default, then the probability of repayment is $qs$. The loan terms are then $R^{IL}(q) = r/(qs)$ and $K^{IL}(q)$ is such that it maximizes

$$
\max_{K} F(K) - R^{IL}K = \max_{K} F(K) - \frac{r}{sq} K.
$$

When there is no strategic default, $q = 1$, the interest rate is $R^{IL}(1) = r/s$ and the loan size is such that $F'(K^{IL}(1)) = r/s$. The borrower’s payoff from repayment is $F(K^{IL}(1))(1 - \varepsilon^{IL}(1))$, where $\varepsilon^{IL}(1)$ is the elasticity of the production function at $K^{IL}(1)$. As before, $\varepsilon^{IL}(1)$ is the share of the output that has to be repaid to the lender when the loan terms assume no strategic default. Differently from Section 3, however, there is a possibility of non-strategic default and, therefore, the loan size and the value of $\varepsilon^{IL}(1)$ depend on $s$. Given the focus on the steady state with no strategic default, I will write $K^{IL}(1), R^{IL}(1)$, and $\varepsilon^{IL}(1)$ without their argument of $q = 1$.

Along the no strategic default steady state, the borrower’s patience must be such that

$$
F(K^{IL}) \leq \frac{1}{1 - s\delta} F(K^{IL})(1 - \varepsilon^{IL}).
$$

Here the LHS is the borrower’s payoff when the current project succeeds, and the borrower strategically defaults. The RHS is the borrower’s payoff when the current project succeeds, and the future payoffs are discounted with $s\delta$. The inequality is satisfied as long as $\delta^{IL} \geq \varepsilon^{IL}/s$.

**Joint Liability (JL):** There are two borrowers with two identical projects $F(\cdot)$ and equal discount coefficients. The probability of success of one project is $s$, and both projects and their outcomes are independent of each other. The two borrowers are jointly liable for their debt. If at least one borrower defaults, then both borrowers are excluded from future loans. Each borrower observes the outcomes of both projects. When his own project fails the borrower has no funds to

---

18Having both borrowers within one group with the same discount is akin to positive-assortative matching among the group members as, presumably, group members have more information about each other’s patience.
contribute to loan repayment. When his own project succeeds, the borrower has to decide whether to repay his own loan and, if the other project fails, whether to repay the second loan.

I focus on the case where if exactly one project fails, the successful borrower can afford to repay both loans (corresponds to Case A in de Quidt et al., 2013), as otherwise there is no positive externality associated with having the second borrower. Assuming no strategic default the probability of repayment both loans is \( s(2 - s) \) and the interest rate, therefore, is \( R^{IL} = \frac{r}{s(2 - s)} \). If the borrower’s project fails his current payoff is zero. If the borrower’s project succeeds, his payoff depends on the success of other borrower’s project. With probability \( s \), the payment is \( R^{IL} K \) and with probability \( 1 - s \), both loans have to be repaid and, therefore, the payment is \( 2R^{IL} K \). The loan size that maximizes the expected repayment payoff is

\[
\max_K F(K) - \left[ sR^{IL} + (1 - s)2R^{IL} \right] K = \max_K F(K) - R^{IL} K,
\]

which implies that \( K^{IL} = K^{JL} \).

The dynamic incentives under the JL-setting have a unique property that is absent in IL. Even though, a borrower is an exponential discounter, when the successful borrower has to repay both loans, his objective becomes that of a hyperbolic discounter. His expected payoff is

\[
\left[ F(K^{IL}) - 2\frac{r}{s(2 - s)} K^{IL} \right] + s \left[ F(K^{IL}) - \frac{r}{s} K^{IL} \right] \delta + \sum_{t=2}^{\infty} s \left[ F(K^{IL}) - \frac{r}{s} K^{IL} \right] \delta^t (s(2 - s))^{t-1}.
\]

For a borrower, the expected discount between periods 0 and 1 is \( 1 - \delta \cdot s \). Since the borrower repays both loans at \( t = 0 \), the probability that the game reaches period 1 is 1; the intertemporal discount is \( \delta \); the probability of the borrower’s success in period 1 is \( s \). The expected discount between periods 0 and \( t \) is \( (s(2 - s))^{t-1} \cdot \delta^t \cdot s \). The probability that the games reaches period \( t \) from period 0, assuming no strategic default, is \( 1 \cdot (s(2 - s))^{t-1} \), where 1 is the probability of reaching period 1 from period 0 and \( (s(2 - s))^{t-1} \) is the probability of reaching \( t \) from period 1. Intertemporal discount is \( \delta^t \). Probability of the borrower’s project success in period \( t \) is \( s \).

Thus, effectively, a borrower contemplating repaying both loans becomes a hyperbolic discounter. For him the discount between periods 0 and 1 is \( \delta s \), and the discount between periods \( t(>0) \) and \( t+1 \), as viewed from period 0, is \( \delta s(2 - s) \). Using the standard \( \beta \cdot \delta \) notations \( \beta = \frac{1}{2 - s} < 1 \) and \( \delta = \delta s(2 - s) \). The source of endogenously arising hyperbolic discounting is similar in its spirit to public goods or, more generally, positive externalities. At \( t = 0 \) the borrower bears the full cost of repaying both loans and moving the game to \( t = 1 \) but does not fully, or rather immediately, internalizes the benefit of doing so. As far as the trade-off between periods \( t = 0 \) and \( t = 1 \) goes,

\[
F(K^{IL}) - 2R^{IL} K^{IL} = F(K^{IL}) - 2\frac{r}{s(2 - s)} K^{IL} = F(K^{IL}) - \frac{2}{2 - s} \cdot R^{IL} K^{IL} = F(K^{IL}) \left( 1 - \frac{2}{2 - s} \varepsilon^{IL} \right)
\]

As long as the elasticity of \( F \) at point \( K^{IL} \) is such that \( \frac{2}{2 - s} \varepsilon^{IL} \leq 1 \), or equivalently \( 2\varepsilon^{IL} \leq 2 - s \), repaying two loans is affordable. A sufficient condition is \( \varepsilon^{IL} \leq 1/2 \), which would hold, for instance, for \( F(K) = K^{\alpha} \) as long as \( \alpha \leq 1/2 \).
there is no benefit from having the second borrower. The trade-off between periods \( t + 1 \) and \( t(> 0) \), as viewed from \( t = 0 \), is different, however. Having the second borrower has a benefit, as it increases the probability of reaching \( t + 1 \), conditional on reaching \( t \). This is why, as long as \( s < 1 \), the borrower discounts period 1 at a heavier rate, i.e. \( \dot{\beta} < 1 \). Note that the logic above holds regardless of whether the stream of future loans is riskless or not. The difference in intertemporal discounts comes from the fact that when the second borrower’s project fails at \( t = 0 \), the benefit of having the second borrower re-appears only at \( t = 1 \).

**Joint Liability versus Individual Liability:** The time inconsistency of JL-borrowers makes the analysis of Section 3 inapplicable. This is why, I restrict the analysis of this section to a comparison no-strategic-default steady states under IL and JL.

**Proposition 4.4.** The JL-borrower will not default along the no-strategic-default steady state as long as
\[
\delta > \delta^{JL} = \frac{2}{s(2 - s)} \frac{\varepsilon(1)}{1 + \varepsilon(1)}.
\]

To compare \( \delta^{JL} \) and \( \delta^{IL} \) note that
\[
\delta^{JL} = \frac{1}{2 - s} \frac{2}{1 + \varepsilon^{IL}} \frac{\varepsilon^{IL}}{s} = \frac{1}{2 - s} \frac{2}{1 + \varepsilon^{IL}} \delta^{IL}.
\]

Therefore, \( \delta^{JL} > \delta^{IL} \) if and only if
\[
\frac{2}{2 - s} \frac{2}{1 + \varepsilon^{IL}} > 1 \iff s > \frac{2\varepsilon(1)}{1 + \varepsilon^{IL}}.
\]

As follows from (7), \( \delta^{JL} \) is greater than \( \delta^{IL} \) for higher \( s \).\(^{20}\) The intuition is that when \( s \) is high, the insurance factor of having the second borrower is less important. Therefore, the borrower has to be more patient in order to be willing to pay off the second loan. Note that it does not matter that when \( s \) is high the probability that the second borrower fails is low. This is because, no matter how small it is, once it occurs the successful borrower should have enough incentives to repay both loans.

With this result, we can now discuss whether non-profit MFIs would prefer IL or JL in our framework. I consider two cases: MFIs maximizing the borrower’s welfare and MFIs maximizing the outreach (Aubert et al., 2009). In my framework, the expected payoff every period is the same in both IL and JL: \( s(F(K^{IL}) - R^{IL}K^{IL}) \). The difference between individual and joint liability steady states is in how many borrowers have access to the loan (if the requirement on \( \delta \) is higher, it can be interpreted as smaller outreach) and the probability of the loan continuation. \( IL \) always has a lower probability that the loan will continue into the next period. This is because the individual probability of success under \( IL \) is lower than the joined probability of success under \( JL \). Thus, a non-profit MFIs who exclusively wants to maximize the borrower’s welfare will always prefer \( JL \).

\(^{20}\)Earlier in footnote 20, we imposed a restriction on \( s \) and \( \varepsilon^{IL} \) to ensure that the borrower can afford to repay both loans. The restrictions was equivalent to \( s < 2 - 2\varepsilon^{IL} \). It is not binding when \( \varepsilon^{IL} < 1/2 \). The two restrictions are mutually compatible, as long as, \( (\varepsilon^{IL})^2 + \varepsilon^{IL} - 1 < 0 \) or \( \varepsilon^{IL} < (\sqrt{5} - 1)/2 \approx 0.62 \). For higher \( \varepsilon^{IL} \), then \( \delta^{IL} \) is always higher.
However, depending on $s$, the JL-loans can be more restrictive, as they require higher $\delta$. When $s$ is high, the extra benefit of insurance is less crucial and, therefore, a higher $\delta$ is needed to make sure that the borrower will repay the groupmate’s loans. Thus, if an MFI’s objective is the number of borrowers served, then such an MFI might opt for the IL scheme in the high $s$ case.

### 4.3 Impossibility of Separating Equilibria

In the main framework it is impossible for more patient types to signal their patience. All types were pooled together, and the only way to reveal high patience is by repaying the loan. In this section, I consider, if it is possible to have an equilibrium with several loans offered, thereby allowing different types to self-select according to their patience.

As before, I assume that lenders’ beliefs about repayment rate fully determine the loan terms, $(K_t, R_t)$. Assume that there are two sequences of loans, one specified by beliefs $\{q_t\}$ and another by beliefs $\{Q_t\}$. Given $\{q_t\}$ and $\{Q_t\}$, the borrower’s strategy is to decide when to default, as well as which loan to use in a given period $t$. Note, that I allow switching between $q$-loans and $Q$-loans, that is a borrower can choose a $q$-loan in period $t-1$, and then upon repayment a $Q$-loan in period $t$. In equilibrium, lenders’ beliefs should be rational given the borrower’s strategy that is $q_t$ (and $Q_t$) should be a correct probability of repayment for a given loan.

**Proposition 4.5.** In any equilibrium $q_t = Q_t$ for every $t$.

**Proof.** Assume not. Let $t \geq 0$ be the first period, when $q_t \neq Q_t$ and without loss of generality assume that $q_t > Q_t$. Two cases are possible. First, no type defaults at $t$, in which case $q_t = Q_t = 1$ which is a contradiction to $q_t \neq Q_t$. Second, some types default. All defaulting types should choose $q$-loan, as it offers a larger loan. But then $Q_t = 1$, which is a contradiction with $q_t > Q_t$.

Note that in the proof I implicitly assumed that each loan is used by a positive mass of types. If no one (or measure zero) uses the $Q$-loan, then it could be possible that $q_t > Q_t$. However, the main insight remains valid. There is no separation of the types in the equilibrium. In every period, either they all use the same loan, or they use different loans but with equivalent terms. Also note that the proof could be easily extended to the case of three or more sequences of loans. For any two loans that are used by a positive measure of borrowers, the terms have to be equivalent.

In the Proposition above, I assumed that the borrower can freely switch between different types of loans. In this sense, labeling loans as being $q$-loans or $Q$-loans can be done in any arbitrary manner. However, the Proposition and its proof remain valid if we do not allow switching between the two sequences of loans. In other words, when a borrower can make a choice in the beginning but then has to stick to the chosen sequence throughout the entire game. The reason why the proof remains valid is because in it I specifically looked at the first period when $q_t \neq Q_t$. Both options are the same before period $t$ and, therefore, all types who default at $t$ will strictly prefer to choose the $q$-sequence, thereby contradicting $q_t > Q_t$.

Finally, the reason why the separation is not possible in my model, is the failure of the single-crossing condition which is the necessary condition for separation. On one hand, as the single-crossing condition requires, a marginal improvement in the future loan terms does lead to a higher
marginal utility of more patient borrowers. However, a marginal improvement of the current loan terms lead to the same increase in marginal utility, regardless of $\delta$. Therefore, the single-crossing condition fails. The proof reflects this theoretical argument. It is impossible for the current period lender to separate types based on the current loan terms. As a consequence, the separation cannot be achieved in any future period either. The failure of single-crossing with respect to $\delta$ is due to an inherent feature of dynamic incentives, which is a reward or punishment of current behavior does not come until the next period. This is why exclusive reliance on dynamic incentives cannot separate borrowers with different discount factors. However, if one can use other instruments, such as collateral, a separation can be possible, as long as the second instrument has an impact at the same period when the default’s decision is made. A quick example is to require a collateral at the amount of $\frac{1}{1 - \varepsilon(1)}(F(K(1)) - R(1)K(1))$ at $t = 0$, prior to receiving any loans. After the collateral is received borrowers are guaranteed the riskless stream of loans, as long as they do not default. Only types with $\delta \geq \varepsilon(1)$, will pay the collateral and will separate themselves from impatient types.

4.4 Separation via Signaling

As the argument above suggests types’ separation via different contracts is impossible in my model. The only way the patience can be revealed is by consistently paying off the loans. Even though it works — in a sense that the patience is eventually revealed — in all equilibria, it leads to smaller than first-best loans, and in all equilibria, but one, it leads to a rapidly decreasing sequence of loans.

In this section, I develop a simple extension of the original model by allowing the borrower to send a costly signal. In general, we would expect poor borrowers in developing countries to lack the ability to send costly signals; both for a lack of funds and a lack of options. Nonetheless, there are documented examples of costly signaling in development literature such as obtaining land titles in Indonesia (Dower and Potamites, 2007). In Indonesia, acquiring land titles is a costly process that is lengthy and very bureaucratic. Even though these titles could be used as collateral, they seem to be used instead as an ex-ante signal of borrower’s creditworthiness. Only 40% of the first-time borrowers use land titles as collateral, and doing so does not improve loan terms beyond the effect of having one.

Formally, the signaling is modeled as follows. Assume that in period $t - 1$, a borrower can experience a signaling shock that enables a borrower to send a signal to lenders. I assume that the timing of the signaling shock is unrelated to $\delta$ and, importantly, whether the borrower sends a signal in a given period or not does not reveal any new information about its patience. This assumption substantially simplifies the analysis, since until the signal is sent, equilibria are defined by $(6)$. For example, in the case of land titles in Indonesia, the timing of getting the title could be beyond the control of the borrower, due to the unpredictability of bureaucratic procedures. Alternatively, the borrower might lack resources to send an informative signal to lenders unless an income shock occurs, which is unrelated to the borrower’s interactions with lenders.

The next proposition provides an example of an equilibrium, where sending a signal credibly
reveals that the borrower is patient and leads to an infinite sequence of riskless loans.

**Proposition 4.6.** There is an equilibrium with signaling where signal $c_{t-1} = (F(K(1)) - F(K_t))\varepsilon(1)$, sent in period $t - 1$ credibly reveals that the borrower is patient.

**Proof.** The equilibrium has the following structure. Until the signal is received, the dynamic unravels according to (6). Upon receiving the signal at period $t - 1$, lenders believe that $q_T = 1$ for every $T \geq t$ and offer the loan $(q, \delta) = (1, \varepsilon(1))$. To ensure that lenders’ beliefs are correct, it has to be the case that types with $\delta < \varepsilon(1)$ will prefer not to send a signal.

Consider a type with $\delta < \varepsilon(1)$. Suppose that without signaling the best time to default is $T \geq t$. Let $\pi_{t,T}(\delta)$ is the profit from default at $T$ from period $t$ point of view. Since $T \geq t$, $\pi_{t,T}(\delta) \leq \pi_{t,t}(\delta) = F(K_t)$.

If type $\delta$ signals at $t - 1$, then upon getting access to an infinite sequence of loans $(1, \varepsilon(1))$, this type will default immediately at period $t$. Its payoff then is $-c_{t-1} + \delta F(K(1))$. If type $\delta$ does not signal and instead defaults at $T$, its payoff (from period $t - 1$ point of view) is $\delta \pi_{t,T}(\delta)$. It is easy to establish that

$$-c_{t-1} + \delta F(K(1)) = -(F(K(1)) - F(K_t))\varepsilon(1) + \delta F(K(1)) < \delta F(K_t) = \delta \pi_{t,t}(\delta) \leq \delta \pi_{t,T}(\delta),$$

or in other words, payoff of type $\delta$ from signaling is lower than from defaulting at $T$. This means that beliefs are correct. That is, if the signal is observed then there will be no default given loan sequence $(1, \varepsilon(1))$.

From the LTM assumption follows that $K_t$ is an increasing function of $q_t$. Thus, more favorable beliefs imply that the signal cost is lower. This observation has an interesting implication. If along the equilibrium the $q$-sequence is not monotone, which is the b)-case of Theorem 3.6, then the best period for a borrower to receive a signaling shock is around the peak of a $q_t$-trajectory. First, when $q_t$ is high, the signal itself is less costly. Second, the borrower receives more capital making the signal more affordable. Thus, signaling even in this simple extension can reduce the inefficiency of equilibria of the original model. Whether the equilibrium is efficient or not when the signaling shock occurs, patient types can credibly signal their patience in exchange for an access to riskless loans.

### 4.5 Alternatives to the Uncertainty over $\delta$ Assumption

In Section 3, I assumed that lenders have uncertainty over the borrower’s discount factor, $\delta$. Uncertainty about $\delta$ implies that lenders are uncertain in regards to how much the borrower values future loans and, therefore, how responsive he is to dynamic incentives. Notably, the value of future loans depends also on factors other than borrower’s discount coefficient. In this section, I introduce alternatives to the uncertainty over $\delta$ assumption, under which results of Section 3 will remain intact. With the exception of the first alternative, the proof will rely on a generalization of the main framework, which is given in Appendix B (Section 7).
Uncertainty over the rate of productivity growth: Assume that $\delta$ is common knowledge, and lenders are uncertain about the rate of productivity growth of the borrower’s production function. That is, assume that the production function grows with rate $A$ so that $F_{t+1}(K) = AF_t(K)$, where $A$ is the borrower’s private information. $A$ is distributed on interval $[A_{\text{min}}, A_{\text{max}}]$, where $\delta A_{\text{max}} < 1$ so that the borrower’s utility is well-defined. If we define $\hat{\delta}$ as $\delta A$, then it becomes identical to the setting in Section 3, where the borrower’s discount coefficient is $\hat{\delta}$, and it is distributed on interval $[\delta A_{\text{min}}, \delta A_{\text{max}}]$.

An important underlying assumption is that lenders cannot learn $A$ in any way other than as described in Section 3. It could be reasonable, for example, if the only information provided by credit bureaus is that the borrower has no prior defaults. However, if the loan size is determined so that to maximize $\max_K F(K_t) - R_t K_t$ and sizes of prior loans are observed, the lenders could use this additional information to deduce $A$.

Uncertainty over proportional collateral: Assume that in the case of default, the borrower looses some fixed portion, $\theta$, of the loan as post-collateral. While lenders might know the monetary value of the collateral, they are uncertain how much the borrower values it. Alternatively, $\theta$ might measure (unobserved by lenders) the borrower’s ability to post-collateral. Thus, the borrower’s profit from payback is, as before, $(1 - \varepsilon_t)F(K_t)$ but the payoff from default is $(1 - \theta)F(K_t)$.

Uncertainty over an outside option: The borrower might have an outside option once lenders restrict the borrower’s access to the credit market. The value of the outside option can be unknown to lenders. Then the payback payoff is $(1 - \varepsilon_t)F(K_t)$, and the default payoff is $F(K_t) + \theta$, where $\theta$ is the NPV of the outside option.

Uncertainty over additive collateral: This example is different from the previous three in that the model with uncertainty over an additive collateral would be different from the one developed in Section 3. Assume that the cost of collateral to the borrower, $\theta$, is additive and is unknown to lenders. Then the payback payoff $(1 - \varepsilon_t)F(K_t)$ and the default payoff is $F(K_t) - \theta$. The reason why the Section 3’s reasoning does not immediately extend to the additive collateral case is that the payoff from default can be negative.

5 Conclusion

Dynamic incentives is a relatively well-understood microfinance methodology. A simple game-theoretical model can readily show that for a patient borrower, the long-term gain from accessing future loans can outweigh the short-term gain from default. Thus, as long as a borrower is suf-

\[21\] An example where lenders could be uncertain about value of collateral is the so-called nontraditional collateral introduced by Bank Rakyat Indonesia. Nontraditional collateral is something that might not have a resale value, but is, nonetheless, valuable to the borrower, such as a domestic animal or land that is not secured by title.
iciently patient, and lenders can punish defaulters by full (or partial) exclusion from the credit market, the borrower will prefer to repay current loans.

In this paper, I develop a new model to study dynamic incentives. I assume that lenders face uncertainty with regards to how much the borrower values future loans and, therefore, how responsive the borrower is to dynamic incentives. The uncertainty about how much the borrower values future loans can occur if lenders are uncertain about the borrower’s patience, outside option, the borrower’s evaluation of post-collateral or the borrower’s productivity growth. Another feature of my model is the impossibility of exclusive contracts between lenders and the borrower. This, in particular, implies that the parties cannot commit to a contract which is longer than one period; and that lenders’ information with regards to a given borrower is symmetric. The latter is the case if there is a credit bureau available to all lenders.

As I show in Section 3, only one equilibrium eliminates the risk of default in the long-run. For the rest of equilibria, all borrowers including the most patient ones default. This highlights the limitations of dynamic incentives. The environment with sufficiently patient types and the full exclusion of defaulters is not enough to prevent patient types from default. Another result of the benchmark model is the drawback to starting small when an exclusive relationship between the borrower and lenders is impossible, and the current loan terms are determined by the anticipated default rate. In my model, starting with smaller loans can be only rationalized by the higher default rate and, as I show in the paper, the eventual default of even most patient types. The only efficient equilibrium has the highest initial loan among all equilibria.

I then study dynamic incentives in the presence of double-dipping. As I show the effect of double-dipping can be ambiguous. Having two loans can not only increase default gains but also repayment gains, especially for credit-constrained borrowers. When default gains are larger, future loans have to be more attractive to generate payoff incentives. In fact, if we compare two equilibria, one with double-dipping and one without, that start with the same initial (single) loan the double-dipping equilibrium will have better future loans and lower default rates. It is important to note that if access to multiple loans occurs unexpectedly for lenders, it will lead to a higher default rate, as observed in the empirical literature.

The final contribution of my paper is an observation that the combination of joint liability and dynamic incentives leads to a framework with endogenously arising hyperbolic discounting. It is an interesting question left to a future research how equilibria in the benchmark model in Section 3 will change once joint liability and endogenous hyperbolic discounting are introduced.

6 Appendix A: Proofs

Proposition 3.2 In any equilibrium \{δt\} is a strictly increasing sequence. The borrower’s optimal strategy \(T(δ)\) is then defined as follows: for types with \(δ \in (δ_t, δ_{t+1})\) it is optimal to default at period \(t + 1\).

Proof. Proof by contradiction. Let \(τ\) be a moment when \(δ_τ > δ_{τ+1}\). Then \(q_{τ+1} = 1\). Indeed, by
definition of $\delta_t$, types with $\delta < \delta_t$ prefer default at $\tau$ to default at $\tau + 1$. Since $\delta > \delta_t$ implies $\delta > \delta_{t+1}$ types with $\delta > \delta_t$ will prefer default at $\tau + 2$ to default at $\tau + 1$. Type with $\delta = \delta_t$ has measure zero.

From (4)

$$\delta_t = \varepsilon(q_t) \frac{F(K(q_t))}{F(K(1))} = \frac{R(q_t)K(q_t)}{F(K(1))}$$

$$\delta_{t+1} = \varepsilon(1) \cdot \frac{F(K(1))}{F(K(q_{t+2}))} = \frac{R(1) \cdot K(1)}{F(K(q_{t+2}))}.$$  

Since $K(q)$ and $R(q)K(q)$ are increasing functions of $q$ and since $q_t$ and $q_{t+2}$ are less than or equal to one, $\delta_t \leq \delta_{t+1}$ which is a contradiction. Furthermore, notice that $\delta_t = \delta_{t+1}$ only if $q_t = q_{t+1} = q_{t+2} = 1$, which in turn would imply $\delta_t = \delta_{t+1} = \varepsilon(1)$. 

**Lemma 6.1.** Assume that $\{\delta_t\}$ is a weakly increasing sequence and assume that $\delta_t < \delta_{t+1}$ for some $t$. Then for any borrower with $\delta$ such that $\delta_t < \delta < \delta_{t+1}$ it is optimal to default at $t + 1$.

This lemma is essentially proved in the main body of the paper. The proof here is repeated for the sake of completeness.

**Proof.** Let $\pi_t(\delta)$ denote the utility of the borrower with discount factor $\delta$ from default at $t$. Then

$$\pi_0(\delta) < \pi_1(\delta) \cdots < \pi_t(\delta) < \pi_{t+1}(\delta) > \pi_{t+2}(\delta) > \ldots.$$  

Indeed, since $\{\delta_t\}$ is weakly increasing $\delta > \delta_t$ implies $\delta > \delta_t$ for any $\tau \leq t$ and, therefore, $\pi_t(\delta) < \pi_{t+1}(\delta)$ for any $\tau \leq t$. Similarly, $\delta < \delta_{t+1}$ implies $\delta < \delta_t$ for any $\tau \geq t + 1$ and, therefore, $\pi_t(\delta) > \pi_{t+1}(\delta)$ for any $\tau \geq t + 1$.  

Finally, I will show that $\{\delta_t\}$ is a strictly increasing sequence. Assume not. Let $\tau$ be the first moment, such that $\delta_t = \delta_{t+1}$. If $\tau = 0$ then $q_0 = 1$, $\delta_0 = \delta_t = \varepsilon(1)$ and from weak-monotonicity $\delta_t \geq \varepsilon(1)$ for every $t$. Since $\varepsilon(1) > \delta_{\min}$, we can use the same logic as in Lemma 3.1 to show that types with $\delta < \varepsilon(1)$ will default at 0, which contradicts $q_0 = 1$. Similarly, if $\tau > 0$ then $\delta_{\tau-1} < \delta_t$ and by Lemma 3.1 types with $\delta \in [\delta_{\tau-1}, \delta_t]$ default at $\tau$ which contradicts $q_t = 1$. The fact that $\{\delta_t\}$ is a strictly increasing sequence together with Lemma 3.1 complete the proof of the Proposition.  

**Proposition 3.3** Let $\{(q_t, \delta_t)\}$ be an equilibrium with initial conditions $(q_0, \delta_0)$ and $\{(Q_t, \Delta_t)\}$ be an equilibrium with initial conditions $(Q_0, \Delta_0)$. If $q_0 > Q_0$ then $q_t > Q_t$ and $\delta_t < \Delta_t$ for every $t$.

**Proof.** Let $q_{t+1}(q, \delta)$ and $\delta_{t+1}(q, \delta)$ denote the result of one iteration of (6) given $(q, \delta)$. Also I will say that pair $(q, \delta)$ is feasible if $0 \leq q \leq 1$ and $\delta_{\min} \leq \delta \leq \delta_{\max}$. The following Lemma does not require $(q, \delta)$ to be a part of an equilibrium trajectory. The only requirement is that one iteration of (6) based on $(q, \delta)$ will result in a feasible $(q_{t+1}, \delta_{t+1})$.

**Lemma 6.2.** Assume that $(q, \delta)$ and $(q_{t+1}, \delta_{t+1})$ are feasible. Then $q_{t+1}$ is an increasing function of $q$ and a decreasing function of $\delta$; $\delta_{t+1}$ is a decreasing function of $q$ and an increasing function of $\delta$. 

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Proof. Pair \((q, \delta)\) defines \(q_{t+1}\) by the first equation of (6) which given the definition of \(\varepsilon(q)\) can be re-written as

\[
F(K(q_{t+1})) = \frac{\varepsilon(q) \cdot F(K(q))}{\delta} = \frac{R(q)K(q)}{\delta}.
\]

Thus \(\frac{\partial q_{t+1}}{\partial \delta} < 0\), and since \(R(q)K(q)\) is an increasing function \(\frac{\partial q_{t+1}}{\partial \delta} > 0\). Given that \(q_{t+1} \leq 1\) from the second equation of (6) it immediately follows that \(\delta_{t+1}\) is a decreasing function of \(q_{t+1}\) and an increasing function of \(\delta\). Together with the established monotonicity of \(q_{t+1}\), this implies that \(\delta_{t+1}\) is a decreasing function of \(q\) and increasing function of \(\delta\). ■

Now we can prove the proposition. Given that along the equilibrium \(q_0 = 1 - \Phi(\delta_0)\), having \(q_0 > Q_0\) implies \(\delta_0 < \Delta_0\). Iteratively applying Lemma 6.2 we get that \(q_t > Q_t\) and \(\delta_t < \Delta_t\) for all \(t\). The requirement of Lemma 6.2 is satisfied because \(\{(q_t, \delta_t)\}\) and \(\{(Q_t, \Delta_t)\}\) are equilibria and, therefore, are feasible. ■

Proposition 3.4

Equilibrium trajectories exist and converge to either the efficient steady state \((1, \varepsilon(1))\) or the no-loan steady state \((0, \delta_{\text{max}})\).

Similarly to the proof of Proposition 3.3 I will denote \(q_{t+1}\) and \(\delta_{t+1}\) as the \(t\)-th iteration of (6) based on initial values \((q, \delta)\). Also I will say that pair \((q, \delta)\) is feasible if \(0 \leq q \leq 1\) and \(\delta_{\text{min}} \leq \delta \leq \delta_{\text{max}}\).

Proof. First, we prove that equilibrium trajectories exist. Let \((Q, \Delta) = (1, \varepsilon(1))\) and \((q_0, \delta_0) = (1 - \Phi(\varepsilon(1)), \varepsilon(1))\) be two initial conditions. Note that while both are feasible, \((1, \varepsilon(1))\) cannot be an equilibrium pair of initial values. This is because we assumed that \(\varepsilon(1) > \delta_{\text{min}}\) and, therefore, \(Q \neq 1 - \Phi(\Delta)\) when \(Q = 1\) and \(\Delta = \varepsilon(1)\).

Pair \((Q, \Delta) = (1, \varepsilon(1))\) is a steady state of (6) and therefore \((Q_{t+1}, \Delta_{t+1})\) are equal to \((1, \varepsilon(1))\) for every \(t\) and are feasible. Given that \(Q > q_0\) and \(\Delta = \delta_0\) from Lemma 6.2 follows that \(q_1 < Q_{t+1} = 1\). Having \(q_1 < 1\) implies that \(\delta_1\) is well-defined, and so the pair \((q_1, \delta_1)\) is feasible. Again, by Lemma 6.2, \(\delta_1 > \Delta_{t+1} = \varepsilon(1)\). Iteratively applying the same argument, we get the feasibility of the trajectory starting with \((q_0, \delta_0)\) which, therefore, is an equilibrium.

Next, I show that in equilibrium \(q\)-sequence is either monotone or eventually becomes monotone. That is if \(q_t > q_{t+1}\) that \(q_s > q_{s+1}\) for any \(s > t\). According to Proposition 3.3, \(q_{t+1}\) is an increasing function of \(q_t\) and a decreasing function of \(\delta_t\) and according to Proposition 3.2, \(\delta_{t+1} > \delta_t\). Therefore, if \(q_t > q_{t+1}\) then \(q_{t+2} < q_{t+1}\). By the same argument one can show that \(q_s > q_{s+1}\) for any \(s > t\). Thus, \(q\)-sequence is either increasing, or eventually becomes decreasing.

Since both \(q\)-sequence and \(\delta\)-sequences are monotone (or eventually monotone) and bounded they converge. Assume that \(\delta_t \to \delta(\neq \delta_{\text{max}})\) then from the second equation of (6) it follows that \(q_t\) should converge to 1, and then from the first equation \(\delta_t \to \varepsilon(1)\). That corresponds to limit point \((1, \varepsilon(1))\). Assume that \(\delta_t \to \delta_{\text{max}}\). If \(q\)-sequence converges to a positive limit, \(\hat{q}\) then the first equation converges to \(\delta_{\text{max}} = \varepsilon(\hat{q})\) which is impossible given our assumption \(\delta_{\text{max}} > \varepsilon(q)\). Thus, \(q_t\) has to converge to zero which corresponds to the second limit point \((0, \delta_{\text{max}})\). ■

\(^{22}\)The reason why \(Q\) variable has subscript ‘+1’ and the \(q\) variable has subscript ‘1’ is just to indicate that the latter is an equilibrium trajectory while the former is simply one iteration of (6).
Proposition 3.5: There exists an equilibrium that converges to \((1, \varepsilon(1))\). This equilibrium is unique.

As before in the proof of this Proposition I will say that pair \((q, \delta)\) is feasible if \(0 \leq q \leq 1\) and \(\delta_{min} \leq \delta \leq \delta_{max}\).

Proof. i) Existence: Consider the set of all trajectories started by \((q_0, \delta_0)\) such that \(q_0 \in [0,1]\) and \(\delta_0 = 1 - \Phi(q_0)\). Let \(A\) be the set of all \(q_0 \in [0,1]\) that generate feasible trajectories and let \(A^c\) be the set of all \(q_0 \in [0,1]\) that generate infeasible trajectories. Both sets are non-empty. \(A\) is non-empty by Proposition 3.4; \(A^c\) is non-empty because \(\varepsilon(1) > \delta_{min}\) and therefore the trajectory that starts at \(q_0 = 1\) is not feasible. From Proposition 3.3 it follows that if \(q_0 \in A\) then all \(Q_0 < q_0\) are in \(A\) and, similarly, if \(q_0 \in A^c\) then all \(Q_0 > q_0\) belong to \(A^c\).

Next we show that \(A\) is an open set. Indeed, take an infeasible trajectory generated by \(q'_0 \in A^c\) and let \(T\) be the first moment such that \(q'_T > 1\). By (6) function \(q_T(q_0)\) is continuous in a neighborhood of \(q'_0\) and therefore \(q_T(q''_0) > 1\) when \(q''_0\) is sufficiently close to \(q'_0\). In other words, for any \(q'_0 \in A^c\) there is a neighborhood of \(q'_0\) that generates infeasible trajectories and so \(A\) is open.

Let \(q_0 = \inf \bar{A}\) and, since \(\bar{A}\) is an open set, \(q_0 \in A\). I claim that the \(q\)-sequence of the dynamic generated by \(\bar{q}\) converges to 1. Assume not, in which case it converges to zero. Consider a period \(T \geq 0\) such that \(\bar{q}_T\) is the maximum of \(\{\bar{q}_t\}\) and \(q_{T+1} < \bar{q}_T\). The feasibility implies that \(\bar{q}_T\) is strictly less than 1.\(^{23}\) For feasible trajectories and fixed \(T\) functions \(q_{T+1}(\cdot)\) and \(q_T(\cdot)\) are continuous functions of the initial value. Thus, by continuity there exists \(q_0 > \bar{q}\) such that \(q_T(q_0) < 1\) for all \(t \leq T\) and such that \(q_T(q_0) > q_{T+1}(q_0)\). The last part implies that the \(q\)-sequence becomes decreasing after \(T\). Which in turn implies that a trajectory that starts with \(q_0\) is also a feasible trajectory since \(q_T(q_0) \leq q_T(q_0) < 1\) for all \(t\). This is a contradiction since \(q_0 > \bar{q}\) and thus should belong to \(\bar{A}\).

ii) Uniqueness: Assume not. Then there are two equilibrium trajectories \((q_1, \delta_1)\) and \((Q_t, \Delta_t)\) that converge to \((1, \varepsilon(1))\). Assume that \(q_0 > Q_0\). Let \(\Psi\) be a function that a feasible pair \((q, \delta)\) into \((q_{+1}, \delta_{+1})\). In other words, \(\Psi\) is an outcome of one iteration of system (6) so that \((q_{+1}, \delta_{+1}) = \Psi(q_t, \delta_t)\). Using Taylor decomposition of \(\Psi\) we get:

\[
\begin{pmatrix}
Q_{t+1} - q_{t+1} \\
\Delta_{t+1} - \delta_{t+1}
\end{pmatrix} 
\approx \Psi\left(\begin{pmatrix}
Q_t - q_t \\
\Delta_t - \delta_t
\end{pmatrix}\right) = 
\begin{pmatrix}
\frac{\partial q_{t+1}}{\partial q_t} & \frac{\partial q_{t+1}}{\partial \delta_t} \\
\frac{\partial q_{t+1}}{\partial \delta_t} & \frac{\partial \delta_{t+1}}{\partial \delta_t}
\end{pmatrix}
\begin{pmatrix}
Q_t - q_t \\
\Delta_t - \delta_t
\end{pmatrix}.
\]

(9)

In Proposition 3.3 it was already established that \(\frac{\partial q_{t+1}}{\partial q_t} > 0\) and \(\frac{\partial q_{t+1}}{\partial \delta_t} < 0\). In fact, we can establish that the latter is bounded away from zero in the neighborhood of \((1, \varepsilon(1))\). Indeed, re-writing the first equation of (6) as \(\delta_T F(K_{t+1}) - \varepsilon(q_t)F(K_t) = 0\) and using the implicit function theorem we get that

\[
\begin{align*}
\frac{\partial q_{t+1}}{\partial \delta_t} &= -\frac{F(K_{t+1})}{\delta_T F'(K_{t+1}) \frac{\partial K(q_{t+1})}{\partial q_{t+1}}} \\
&< \nu < 0,
\end{align*}
\]

\(^{23}\)Assume there exists feasible trajectories such that \(q_t = 1\) for some \(t\). This would imply then that \(\delta_t = \delta_{t-1}\), which violates Proposition 3.2.
for some small negative \( \nu \). Such \( \nu \) exists because \( K(\cdot) \) is assumed to be continuously differentiable
and, therefore, \( K'(\cdot) \) is bounded from infinity.

Re-writing the second equation of (6) as \( \Phi(\delta_{t+1}) - 1 + q_{t+1}(1 - \Phi(\delta_t)) = 0 \) and using the implicit function theorem we get that
\[
\frac{\partial \delta_{t+1}}{\partial q_t} = -\frac{(1 - \Phi(\delta_t)) \frac{\partial q_{t+1}}{\partial q_t}}{\phi(\delta_{t+1})},
\]
and
\[
\frac{\partial \delta_{t+1}}{\partial \delta_t} = q_{t+1} \frac{\phi(\delta_t)}{\phi(\delta_{t+1})} - \frac{(1 - \Phi(\delta_t)) \partial q_{t+1}}{\partial \delta_t}.
\]
The first expression is always non-positive because \( \frac{\partial q_{t+1}}{\partial q_t} > 0 \). The second expression consists of two terms. Given that \( \varepsilon(1) < \delta_{\text{max}} \), the first term converges to 1 as \( (q_t, \delta_t) \) converges to \( (1, \varepsilon(1)) \). Given that \( \partial q_{t+1}/\partial \delta_t \) is bounded away from 0 the second term is negative and is also bounded away from zero. Therefore, in the neighborhood of \( (1, \varepsilon(1)) \), we have that \( \frac{\partial \delta_{t+1}}{\partial \delta_t} > 1 + \mu \) for some \( \mu > 0 \).

Given established inequalities together with the fact that \( Q_t < q_t \) and \( \Delta_t > \delta_t \), we have from (9) that
\[
\Delta_{t+1} - \delta_{t+1} = \frac{\partial \delta_{t+1}}{\partial q_t} \cdot (Q_t - q_t) + \frac{\partial \delta_{t+1}}{\partial \delta_t} \cdot (\Delta_t - \delta_t) > \Delta_t - \delta_t
\]
when both trajectories are close to \( (1, \varepsilon(1)) \). Therefore, there cannot be two trajectories converging to the efficient outcome. \( \blacksquare \)

**Proposition 4.3:** Let \( K(q) \) and \( R(q) \) satisfy the LTM. Assume also that \( F'(K)K \) is an increasing function of \( K \) and \( R(q) \) a decreasing function of \( q \). Then \( \max_{K \in [K(q),2K(q)]} F(K) - R(q)K \) and \( F(2K(q)) - \max_{K \in [K(q),2K(q)]} F(K) - R(q)K \) are increasing functions of \( q \).

**Proof.** First consider the case when a double-dipping borrower is credit-constrained, that is \( F'(2K(q)) \geq R(q) \). Then the gains from default are \( 2R(q)K(q) \), which is an increasing function of \( q \) as \( K(q) \) and \( R(q) \) satisfy the LTM. The repayment payoff is \( F(2K(q)) - 2R(q)K(q) \). Its derivative with respect to \( q \) is equal to \( 2F'(2K(q))K'(q) - 2R'(q)K(q) - 2R(q)K'(q) \). It is positive because \( F'(2K(q)) \geq R(q) \) and \( R'(q) < 0 \).

Next, consider the case when a double-dipping borrower is not credit-constrained and the optimal loan \( K^* < 2K(q) \). Then \( F'(K^*) = R(q) \) and the repayment payoff is \( F(K^*) - R(q)K^* \). By the envelope theorem its derivative with respect to \( q \) is \( -R'(q)K^* > 0 \). Thus the repayment payoff satisfies the LTM. The gain from default is
\[
F(2K(q)) - \max_{K \in [K(q),2K(q)]} \{ F(K) - R(q_t)K \} = F'(K^*)K^* + F(2K(q)) - F(K^*).
\]
Taking the derivative with respect to \( q \) we have
\[
F''(K^*)K^*(K^*)_q + F'(K^*)(K^*)_q + F'(2K)(2K)_q - F'(K^*)(K^*)_q = F''(K^*)K^*(K^*)_q + F'(2K)(2K)_q.
\]
Applying the implicit function theorem to the FOC, \( F'(K^*) - R(q) = 0 \), we get that \( (K^*)_q' = R'(q)/F''(K^*) \) and so (10) can be re-written as
\[
R'(q)K^* + F'(2K)(2K)_q'.
\]
By the LTM \((RK)^\prime_\prime > 0\) and, therefore, \(R'(q)K(q) + R(q)K'(q) > 0\). Then,
\[
R'(q)K^* + F'(2K)(2K)_q' = K'(q) \left( \frac{R'(q)}{K'(q)} K^* + F'(2K) \cdot 2 \right) > K'(q) \left( -\frac{R(q)}{K(q)} K^* + F'(2K) \cdot 2 \right) \\
= \frac{K'(q)}{K(q)} (R(q)K^* + F'(2K) \cdot 2K(q)) > 0.
\]

In the last step I used that \(F'(K^*) = R(q)\), \(K^* < 2K(q)\) and that \(F'(K)K\) is an increasing function of \(K\).

**Proposition 4.4:** Consider two models, one with and one without double-dipping, that are based on the same \(K(q)\) and \(R(q)\). Then

i) The efficient steady state in equilibrium with double-dipping requires higher patience, i.e. \(\delta_{\text{dd,eff}} > \delta_{\text{sd,eff}}\).

ii) Assume that loan terms are such that a double-dipping borrower is credit-constrained, i.e. \(F'(2K(q)) \geq R(q)\) for every \(q\). If equilibrium trajectories under SD and DD begin with the same initial \(q_0\) then \(q_{t\text{dd}}^d > q_{t\text{sd}}^d\) and \(\delta_{t\text{dd}}^d < \delta_{t\text{sd}}^d\) for every \(t > 0\).

iii) Assume a double-dipping borrower is not necessarily credit-constrained. Assume also that \(F'(K)K\) is an increasing function of \(K\), while \(F'(K)K/F(K)\) is a weakly decreasing function of \(K\). Then if equilibrium trajectories under SD and DD begin with the same initial \(q_0\) then \(q_{t\text{dd}}^d > q_{t\text{sd}}^d\) and \(\delta_{t\text{dd}}^d < \delta_{t\text{sd}}^d\) for every \(t > 0\).

**Proof.** i) The discount factors of a borrower indifferent between the default and repaying on the stream of riskless loans with and without double-dipping are
\[
\delta_{\text{dd,eff}} = \frac{F(2K(1)) - (F(K^*(1)) - rK^*(1))}{F(2K(1))} \quad \text{and} \quad \delta_{\text{sd,eff}} = \frac{rK(1)}{F(K(1))} (= \varepsilon(1)),
\]
where \(r = R(1)\) and \(K^*(1)\) is the the optimal loan size under DD given the riskless interest rate.

Function \(F(\cdot)\) is concave, \(F(0) = 0\) and, therefore, \(K/F(K)\) is increasing. Since \(2K(1) \geq K^*(1) \geq K(1)\), we have that
\[
\delta_{\text{dd,eff}} F(2K(1)) = F(2K(1)) - (F(K^*(1)) - rK^*(1)) \\
\geq F(2K(1)) - F(2K(1)) \left( 1 - \frac{rK^*(1)}{F(K^*(1))} \right) = F(2K(1)) \frac{rK^*(1)}{F(K^*(1))} \\
\geq F(2K(1)) \frac{rK(1)}{F(K(1))} = \delta_{\text{sd,eff}} F(2K(1)).
\]
The first inequality is strict unless \(K^*_t = 2K(1)\) and the second inequality is strict unless \(K^*_t = 2K(1)\). Thus, \(\delta_{\text{dd,eff}} > \delta_{\text{sd,eff}}\).

ii) Consider SD- and DD-equilibria that start with the same \(q_0\) and, therefore, the same \(\delta_0\). I will show that \(q_{t\text{dd}}^d > q_{t\text{sd}}^d\) and then a similar argument can be extended to any \(t\). Let \(K_0 = K(q_0)\) and \(R_0 = R(q_0)\) and let \(k^d\) be a solution to
\[
\delta F(k^sd) = R_0 K_0.
\]

(11)
Equation (10) implicitly defines $k^{sd}$ as a function of $\delta$. Note that $k^{sd}(\delta_0) = K_1^{sd}$. Similarly, for $\alpha \geq 1$ let $k^{dd}$ be a solution to

$$\delta F(\alpha k^{dd}) = F(\alpha K_0) - (F(K_0^*) - R_0 K_0^*),$$

where $K_0^*$ is the optimal loan size given $R_0$ on the interval $[K_0, \alpha K_0]$. (11) implicitly defines $k^{dd}$ as a function of $\alpha$ and $\delta$. Note that, $k^{dd}(1, \delta_0) = (R_0 K_0) / \delta_0 = K_1^{sd}$ and $k^{dd}(2, \delta_0) = K_1^{dd}$.

As long as $\alpha$ is such that $K_0^* = \alpha K_0$, that is the borrower is credit-constrained then $k^{dd} > k^{sd}$. Indeed,

$$\delta F(\alpha k^{dd}) = F(\alpha K_0) - (F(K_0^*) - R_0 K_0^*) = \alpha R_0 K_0 = \alpha \delta F(k^{sd}) > \delta F(\alpha k^{sd}),$$

where the last inequality follows from concavity of $F(\cdot)$ and $F(0) = 0$. Thus if a double-dipping borrower remains to be credit-constrained, i.e. $K_0^* = 2 K_0$, then the single DD-loan in period 1 will be higher than the SD-loan. From $K_1^{dd} > K_1^{sd}$ it follows that $q_1^{dd} > q_1^{sd}$ and therefore, $\delta_1^{dd} < \delta_1^{sd}$. By iteratively applying the logic above one can show that $q_t^{dd} > q_t^{sd}$ and $\delta_t^{dd} < \delta_t^{sd}$ for every $t$. This proves ii).

iii) When a double-dipping borrower is no longer credit-constrained the outcome depends on $F(\cdot)$. Let $\hat \alpha$ be such that $F'(\hat \alpha K_0) = R_0$ and $1 \leq \hat \alpha < 2$. From the argument above $k^{dd}(\hat \alpha, \delta_0) > k^{sd}(\delta_0)$. Thus if $k^{dd}$ is an increasing function of $\alpha$ then $K_1^{dd} = k^{dd}(2, \delta_0) > k^{sd}(\delta_0) = K_1^{sd}$.

Applying the implicit function theorem to (11) and taking into account that $K_0^*$ is a constant when $\alpha > \hat \alpha$ we get

$$\frac{\partial k^{dd}}{\partial \alpha} = - \frac{\delta k^{dd} F'(\alpha k^{dd}) - K_0 F'(\alpha K_0)}{\delta \alpha F'(\alpha k^{dd})}.$$  

(13)

The denominator is positive. From (11) we have $k^{dd} < K_0$ when $\delta = 1$. Thus, when $\delta = 1$ and $K'F(K)$ is an increasing function then the numerator is negative and the entire derivative is positive. Term $\delta k^{dd} F'(\alpha k^{dd})$ is a weakly increasing function of $\delta$ as long as $KF'(K)/F(K)$ is a weakly decreasing function of $K$. Therefore, the numerator of (12) is negative for every $\delta \leq 1$ and $k^{dd}$ is an increasing function of $\alpha$ for every $\delta \leq 1$. This proves that $K_1^{dd} > K_0^{dd}$.

Just like in the proof of ii) from $K_1^{dd} > K_1^{sd}$ follows that $q_1^{dd} > q_1^{sd}$ and, therefore, $\delta_1^{dd} < \delta_1^{sd}$. By iteratively applying the logic above one can show that $q_t^{dd} > q_t^{sd}$ and $\delta_t^{dd} < \delta_t^{sd}$ for every $t$. 

**Proposition 4.6:** The JL-borrower will not default along the no-strategic-default steady state as long as $\delta > \delta^{IL} = \frac{2}{s(2 - s)} \frac{\varepsilon^{IL}}{1 + \varepsilon^{IL}}$.

**Proof.** Under JL, there is no strategic default as long as the borrower is willing to repay both loans, which is when the borrower’s project succeeds and his groupmate’s project does not. Since

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24 In what follows, there is nothing special about 2. If the borrower has access to $N$ lenders, what matters is whether the borrower with $N$ loans is credit-constrained or not.

25 From (11), $\delta F(k^{dd})$ does not depend on $\delta$. Thus, $k^{dd}$ is a decreasing function of $\delta$. Re-write $\delta k^{dd} F'(\alpha k^{dd})$ as $\delta F(k^{dd}) k^{dd} F'(k^{dd}) / F(k^{dd})$. The first term does not depend on $\delta$ by (11), and the fraction term is a weakly decreasing function of $k^{dd}$ and, therefore, is a weakly increasing function of $\delta$. 

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33
the loan stream is constant it is sufficient to guarantee that the borrower does not default at \( t = 0 \). As the borrower’s problem is equivalent to hyperbolic discounter’s problem, there is no default at \( T = 0 \) as long as default at \( T = 1 \) or \( T = \infty \) is more profitable. Using definitions of \( \hat{\beta} = \frac{1}{2 - s} \) and \( \hat{\delta} = \delta(2 - s) \) the borrower’s payoff from default at \( T = \infty \) is

\[
F(K^{IL}) - 2\frac{r}{s(2 - s)}K^{IL} + \hat{\beta} \sum_{t=1}^{\infty} \hat{\delta}^t \left[ F(K^{IL}) - \frac{r}{s} K^{IL} \right] = F(K^{IL}) - 2\frac{r}{s(2 - s)}K^{IL} + \hat{\beta} \frac{\hat{\delta}}{1 - \hat{\delta}} \left[ F(K^{IL}) - \frac{r}{s} K^{IL} \right].
\]

The payoff from default at \( T = 0 \) is \( F(K^{IL}) \). The former is larger if

\[
\hat{\beta} \frac{\hat{\delta}}{1 - \hat{\delta}} (1 - \varepsilon^{IL} F(K^{IL})) > 2\frac{r}{s(2 - s)} K^{IL} = \frac{2}{2 - s} \varepsilon^{IL} F(K^{IL}).
\]

Since \( \hat{\beta} = 1/(2 - s) \) this is equivalent to

\[
\frac{\hat{\delta}}{1 - \hat{\delta}} (1 - \varepsilon^{IL}) > 2\varepsilon^{IL} \iff \hat{\delta} > \frac{2\varepsilon^{IL}}{1 + \varepsilon^{IL}} \iff \delta > \frac{2}{s(2 - s)} \varepsilon^{IL}. \quad (14)
\]

Comparing defaults between \( T = 1 \) and \( T = 0 \) we get that the former is more profitable as long as

\[
F(K^{IL}) - 2\frac{r}{s(2 - s)}K^{IL} + s\delta \left[ F(K^{IL}) \right] \geq F(K^{IL}),
\]

which, given \( \frac{r}{s} K^{IL} = \varepsilon^{IL} F(K^{IL}) \) is equivalent to

\[
\delta \geq \frac{2}{s(2 - s)} \varepsilon^{IL}. \quad (15)
\]

Clearly, if (12) is satisfied then so is (13). Therefore, the borrower will not default as long as (12) is satisfied. □

7 Appendix B: A Model with Generalized Payoff Functions and an Uncertainty Source

In this Appendix, I develop a generalized version of the Section 3 framework. The generalization will be in terms of the borrower’s payoff functions and the source of uncertainty. I will introduce five assumptions that are sufficient to maintain the results of Section 3. Then I will verify whether the alternatives introduced in Section 4.3 satisfy those assumptions. Given the simplicity of alternatives from Section 4.3, developing a generalized payoff function model might seem an overkill. However, it helps to pinpoint the exact assumptions that are needed for my results.
Model with generalized Payoff Functions: Assume that $\delta$ is common knowledge but that there is a heterogeneity among borrowers that is captured by parameter $\theta$, and it is unobserved by lenders. Each period a borrower has two possible actions: default, $d$, or pay back, $p$. The borrower’s one-period profit from either of the two actions depends on two parameters: $q_t$ and $\theta$. I denote the default profit as $\pi_d(q_t, \theta)$ and payback profit as $\pi_p(q_t, \theta)$ and assume that both functions are continuous.

The borrower’s utility from default at period $T$ is

$$\sum_{t=0}^{T-1} \delta^t \pi_p(q_t, \theta) + \delta^T \pi_d(q_T, \theta).$$

The indifference condition between default at period $t$ and period $t+1$ can be written as

$$\pi_p(q_t, \theta) + \delta \pi_d(q_{t+1}, \theta) = \pi_d(q_t, \theta),$$

or equivalently

$$\delta \pi_d(q_{t+1}, \theta) = \pi_d(q_t, \theta) - \pi_p(q_t, \theta).$$

Term $\pi_d - \pi_p$ represents the borrower’s gain from default, net of what could be earned by repaying the loan. The term on the left represents the default profit in the next period. Note that since $\delta$ is common knowledge it is unobserved $\theta$ that generates uncertainty over the borrower’s trade-off between default at $t$ and $t+1$.

I make the following assumptions on $\theta$ and profit functions $\pi_d, \pi_p$ and $\pi_d - \pi_p$:

A1. $\pi_d(q_t, \theta), \pi_p(q_t, \theta)$ and $\pi_d(q_t, \theta) - \pi_p(q_t, \theta)$ are positive when $q_t > 0$ and zero when $q_t = 0$.

A2. $\pi_d(q_t, \theta), \pi_p(q_t, \theta)$ and $\pi_d(q_t, \theta) - \pi_p(q_t, \theta)$ are strictly increasing functions of $q_t$;

A3. Partial derivatives of $\pi_p$ and $\pi_d$ with respect to $q$ are finite;

A4. If the borrower with $\theta^0$ is indifferent between default at $t$ and $t+1$, then all borrowers with $\theta > \theta^0$ and only they prefer default at $t+1$ to default at $t$.\textsuperscript{26} A sufficient condition is:

$$\frac{\partial \pi_p(q_t, \theta)}{\partial \theta} + \delta \frac{\partial \pi_d(q_{t+1}, \theta)}{\partial \theta} > \frac{\partial \pi_d(q_t, \theta)}{\partial \theta}$$

A5. Support of $\theta$ is an interval $I = [\theta_{\min}, \theta_{\max}]$ such that $\pi_p(1, \theta_{\min}) + \delta \pi_d(1, \theta_{\min}) < \pi_d(1, \theta_{\min})$ and $\pi_p(q, \theta_{\max}) + \delta \pi_d(q, \theta_{\max}) > \pi_d(q, \theta_{\max})$ for every $q$.

Assumption A1 states that profits from paying back as well as gains from default, $\pi_d - \pi_p$ are always positive. Assumption A2 states that higher confidence means higher gains after repaying, and higher gains from default. Assumption A3 is a technical assumption that is used for Proposition 3.5 (the uniqueness part). Assumption A4 is based on the premise that borrowers with higher $\theta$

\textsuperscript{26}An alternative reformulation of A4 is if $\theta^0$ is indifferent between default at $t$ and $t+1$ then all borrowers with $\theta < \theta^0$ and only they prefer default at $t+1$ to default at $t$.

35
are more likely to postpone default until the next period, which corresponds to the case of more patient borrowers in the main model. Finally, A5 implies that borrowers with low $\theta$ will default even on the most favorable loans, whereas borrowers with the highest $\theta$ will postpone the default when offered the same loan terms at $t$ and $t+1$.

Together, Assumption A4 and A5 guarantee that there exists a unique $\theta^0$ such that $\theta_{\min} < \theta^0 < \theta_{\max}$ that solves the equation $\pi_p(1, \theta^0) + \delta \pi_d(1, \theta^0) = \pi_d(1, \theta^0)$. Comparing to the main model $\theta^0$ is the analogue of $\varepsilon(1)$. It is the type that is indifferent between default and not when offered a sequence of riskless loans.

All other assumptions are the same as in the main framework. We can define $\theta_t$ as the type that is indifferent between default at $t$ and $t+1$. System (6) becomes

$$
\begin{cases}
\delta \pi_d(q_{t+1}, \theta_t) = \pi_d(q_t, \theta_t) - \pi_p(q_t, \theta_t) & t \geq 0 \\
q_{t+1} = \frac{1 - \Phi(\delta_{t+1})}{1 - \Phi(\delta_t)} & t \geq -1
\end{cases}
$$

and it unravels in a similar manner. Given initial values $(q, \theta)$ we can solve for $q_{t+1}$ from the first equation, and then if $q_{t+1} \leq 1$ use the second equation to calculate $\delta_{t+1}$. Note that when $q_{t+1} = 0$ the LHS of the first equation is 0, while the RHS is positive. This means that $q_{t+1}$ has to be positive. As in the case of the main framework it is possible, however, that there is no solution $q_{t+1}$ on interval $[0, 1]$, in which case, the trajectory becomes infeasible. The interpretation is standard, there is no loan in the next period that is high enough to prevent default at the current period.

First, I check whether alternatives from Section 4.3 satisfy the generalized framework. Then I will prove that as long as assumptions A1 through A5 are satisfied, all the results from Section 3 are applicable to a model with generalized payoff functions.

- **Uncertainty over the rate of productivity growth:** The framework where lenders face uncertainty over the rate of productivity growth is not an immediate example of the generalized model developed in this Appendix. This is because $\pi_d$ and $\pi_p$ change with time due to the productivity growth. In Section 4.3, it was shown how to reformulate it so that results of Section 3 hold.

- **Uncertainty over proportional collateral:** In this case $\pi_p = (1 - \varepsilon_t)F(K_t)$ and $\pi_d = (1 - \theta)F(K_t)$. The indifference condition becomes

$$
\delta F(K_{t+1})(1 - \theta_t) = F(K_t)(\varepsilon_t - \theta_t).
$$

Assumptions A1 through A5 are satisfied as long as $\theta_t < \varepsilon_t$. Assumption A4 is satisfied because $\delta F(K_{t+1}) < F(K_t)$, as follows from (15). Therefore if type $\theta_t$ is indifferent between default at $t$ and $t+1$ then all higher types will prefer to postpone the default. Inequality $\theta < \varepsilon_t$ is needed for assumption A5. Otherwise nobody strategically defaults, however, the no-default outcome is achieved through the use of collateral and not via dynamic incentives. Thus, we have the following conclusion. If $\theta > \varepsilon_t$, then no default will occur and therefore high collateral is enough to stop strategic defaults. The combination of small collateral
and dynamic incentives, however, is plagued with the same problem as just using dynamic incentives.

- **Uncertainty over an outside option:** In this case the payback payoff is $\pi_p = (1 - \varepsilon_t)F(K_t)$ and the default payoff is $\pi_d = F(K_t) + \theta$, where $\theta > 0$ is an outside option. The indifference condition is

$$\delta(F(K_{t+1}) + \theta) = \varepsilon_t F(K_t) + \theta$$

Since having a higher outside option makes borrower more willing to default, one should use an alternative to A4, as explained in the footnote 27, which is that it is types with lower $\theta$ that prefer to wait until $t + 1$.

- **Uncertainty over additive collateral:** As mentioned Section 3 cannot be extended to this case. Here $\pi_p = (1 - \varepsilon_t)F(K_t)$ and the default payoff is $\pi_d = F(K_t) - \theta$. This modification does not satisfy A1 as one-stage profit can be negative. It is more than a minor technical detail. Recall, that in the original model $q_{t+1}$ was guaranteed to be non-negative and so to ensure the feasibility it was enough to keep $q_t$ to be less than one, which is no longer the case.

**Proposition 7.1.** If Assumptions A1 through A5 are satisfied then Theorem 3.6 holds. In the case c) of Theorem 3.6, the sequence $\{\theta_t\}$ converges to $\theta^0$.

**Proof.** The proof will be simply re-stating the proofs of the propositions from Section 3. Given the sequence of lenders’ beliefs $\{q_t\}$, $\theta_t$ is determined from indifference condition $\delta \pi_d(q_{t+1}, \theta_t) = \pi_d(q_t, \theta_t) - \pi_p(q_t, \theta_t)$. First, I show that in equilibrium this indifference condition will determine $\theta_t$ that is between $\theta_{\min}$ and $\theta_{\max}$. By A5, in equilibrium $\theta_t > \theta_{\min}$. Otherwise $q_t = 1$, and since $q_{t+1} \leq 1$ for type $\theta_{\min}$, the payoff from immediate default is higher. By A4, the LHS has higher derivative with respect to $\theta$ than the RHS and therefore, $\theta_t$ must be greater than $\theta_{\min}$. Similarly, in equilibrium $\theta_t < \theta_{\max}$ for any $t$. This is because if $\theta_t > \theta_{\max}$, then for every type default at $t$ is better than at $t + 1$. Then it means that no one would default at $t + 1$. Then $q_{t+1} = 1$, in which case by A5 the type with $\theta_{\max}$ (as well as types with $\theta$ close to $\theta_{\max}$) would prefer to postpone the default until $t + 1$, which is a contradiction to $\theta > \theta_{\max}$.

Next, as in Proposition 3.2, we can establish that in equilibrium the $\theta$-sequence is strictly increasing. Weak monotonicity of $\theta$ follows from A2 and A4. Assume not. If $\theta_{t+1} < \theta_t$ then $q_{t+1} = 1$ (from A4). Furthermore, for type $\theta_{t+1}$ default at $t$ is better than at $t + 1$:

$$\delta \pi_d(1, \theta_{t+1}) < \pi_d(q_t, \theta_{t+1}) - \pi_p(q_t, \theta_{t+1}).$$

By definition of $\theta_{t+1}$ this type is indifferent between default at $t + 1$ and $t + 2$ and so

$$\delta \pi_d(q_{t+2}, \theta_{t+1}) = \pi_d(1, \theta_{t+1}) - \pi_p(1, \theta_{t+1}).$$

27 Note that by a standard backward induction argument there cannot be period $T$ such that at or before $T$ all types default. Which means that types with $\theta$ close to $\theta_{\max}$ will stay in the game.
The two expressions contradict each other, since by A2 the LHS and RHS are strictly increasing functions of $q$. However, as it was the case in Proposition 3.2 it is possible to have $\theta_t = \theta_{t+1}$ but if and only if $q_t = q_{t+1} = q_{t+2} = 1$ and $\theta_t = \theta_{t+1} = \theta^0$, where recall that $\theta^0$ is an analogue of $\varepsilon(1)$ in the main model and is the solution to $\pi_p(1, \theta^0) + \delta \pi_d(1, \theta^0) = \pi_d(1, \theta^0)$. Lemma 3.1 immediately follows from A4. By using A4 and A5 we can prove strict monotonicity of $\theta$ in the same manner as in Proposition 3.2.

Rewriting the proof for the remaining propositions is fairly straightforward. Lemma 6.2, which is the central element of Proposition 3.3 follows from A2 (regarding $\partial q_{t+1}/\partial q_t > 0$) and A4 with A2 (regarding $\partial q_{t+1}/\partial \theta_t < 0$).

The existence of equilibria (the first part of Proposition 3.4) follows from Lemma 6.2, which was proved within the Proposition 3.3 and A1, where in the proof instead of using $\varepsilon(1)$ one should use $\theta^0$. Assumption A1 guarantees that $q_{t+1}$ is positive and the trajectory $(1, \theta^0)$ prevents $q$ from going above 1.

The two limit points of system (14) are $(0, \theta_{\max})$ and $(1, \theta^0)$. Note that A4 and A5 together imply that there exists $\theta^0 \in I$ such that $\pi_p(1, \theta^0) + \delta \pi_d(1, \theta^0) = \pi_d(1, \theta^0)$, and therefore $(1, \theta^0)$ is a steady state. The convergence of equilibrium dynamic to one of the limit points comes from the fact that if $q_t > q_{t+1}$ then $q_s > q_{s+1}$ for any $s > t$. That again, would follow from Lemma 6.2. Since both $q$- and $\delta$-sequences are bounded and monotone or eventually monotone they converge.

Proposition 3.5 (existence part) follows from already established facts that an equilibrium exists, that equilibrium trajectories converge to one of the steady states, A5, continuity, and Proposition 3.2 (monotonicity of $\theta_t$). Finally, the uniqueness part of Proposition 3.5 uses a technical assumption A3. That $\partial q_{t+1}/\partial q_t > 0$ and $\partial q_{t+1}/\partial \theta_t < 0$ comes from Proposition 3.3. The other two inequalities ($\partial \theta_{t+1}/\partial q_t < 0$ and $\partial \theta_{t+1}/\partial \theta_t > 1 + \mu$) come from the second equation of (6) which did not change.

The result regarding the equilibrium comparison from the borrower’s point view, that is that borrowers prefer equilibria with higher $q_0$ remains valid. It follows from Proposition 3.3 and A2 (as borrowers prefer loans with higher $q$, whether they default or not).

References


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