Grade Inflation and Education Quality

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Abstract
We consider a game in which schools compete to place graduates in two distinct ways: by investing in the quality of education, and by strategically designing grading policies. In equilibrium, schools issue grades that do not perfectly reveal graduate abilities. This leads evaluators to have less-accurate information when hiring or admitting graduates. However, compared to fully-revealing grading, strategic grading motivates greater investment in educating students, increasing average graduate ability. Allowing grade inflation and related grading strategies can increase the probability that evaluators select high-ability graduates.

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Grades A and B are sometimes given too readily – Grade A for work of no very high merit, and Grade B for work not far above mediocrity. ... One of the chief obstacles to raising the standards of the degree is the readiness with which insincere students gain passable grades by sham work.

–Report of the Committee on Raising the Standard, Harvard University, 1894

1 INTRODUCTION

Since the 1980s, the mean grade point average at American colleges and universities has risen at a rate between 0.1 and 0.15 points per decade. Most of this increase can be attributed to an increase in the share of As assigned, and a decrease in the share of grades assigned at the low end of the scale. The central concern is that these effects are driven by a drop in grading standards, rather than enhanced student performance or ability. Changes in grading standards can have important consequences: when high grades are assigned liberally, they convey less information to employers, graduate schools, and other evaluators about a student’s true ability and lead to less-informed placement decisions.

The majority of research on grade inflation in the economics literature (surveyed below) establishes that noisy grading policies are a natural consequence of strategic interactions between schools, arising in equilibrium as schools compete to place their graduates. Grade inflation may therefore be a more fundamental phenomenon than popular wisdom suggests. In addition, the literature has consistently documented the welfare costs resulting from grade inflation and other noisy grading strategies. In many of these analyses, however, schools only reveal information about student ability, doing nothing to improve it. This is in contrast to much of the education literature, which shows that certain school investments—for example, recruiting more-effective teachers—improve graduate ability.

Our analysis contributes to the literature by considering the interaction between a school’s investment in education quality and its choice of grading policy. We show that the negative welfare implications established by the economics literature (and generally taken for granted) are often reversed when a school’s investment in education is accounted for. Allowing schools the freedom to strategically choose grading policies changes the incentives for schools to invest in developing student ability. In equilibrium, strategic grading leads to greater investment by schools. Although transcripts are less-informative, the average ability of graduates is higher.

We consider a three-stage model of school competition. In the first stage, schools simultaneously invest in education quality, which determines the probability that they produce a high-ability graduate. In the second stage, schools simultaneously design grading policies. These grading policies determine how transcripts are assigned to students of different abilities and affect the inferences that employers, graduate schools, and other evaluators make when observing a student’s transcript. In the third stage, each school produces a graduate who is then evaluated by a third party. The evaluator benefits if she selects a high-ability graduate to receive a prize (e.g., job, promotion, school admission or fellowship). Observing investments, grading policies, and transcripts, she assigns the prize to the graduate she believes is more likely to be high ability.

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1We found this quotation in Kohn (2002).
2See data compiled by Rojstaczer (2011).
3Zubrickas (2010) and Dubey and Geanakoplos (2010) are notable exceptions, which we discuss below.
4Rivkin, Hanushek and Kain (2005) demonstrate that teachers have powerful effects on student achievement and test scores. At the same time, easily observable teacher characteristics like education and experience explain very little of the variation in teacher quality. Thus, identifying, recruiting, and retaining high-quality teachers is a costly investment, with the potential to significantly improve student outcomes. Other types of investments include caps on class sizes, providing opportunities for teacher professional development, increasing the allocation of school resources, and better monitoring and incentives for student effort.
Throughout the paper, we compare equilibrium outcomes when schools grade strategically with outcomes arising in a “fully-revealing” benchmark. In this benchmark, all uncertainty about graduate ability is resolved during evaluation; this is often viewed as the ideal in policy discussion. We consider means by which the fully-revealing benchmark may be approximated in Section 3.1.

In the fully-revealing benchmark, the only way for a school to increase the probability the evaluator selects its graduate is to provide him a better education. It does this by investing more in school quality. The same incentives exist when schools choose grades strategically; however, an additional incentive for investment emerges. When no restrictions are placed on grading, both schools choose equilibrium grading policies that distort grades of both low- and high-ability graduates. By assigning a particular transcript to both types of students, the school makes a low-ability graduate with this transcript appear to be better than he truly is. The higher the likelihood that a particular transcript is assigned to a high-ability student, the better the evaluator’s inference about any graduate receiving this transcript. The school that invested less in education quality in the first stage is less likely to produce a high-ability graduate, and is therefore limited in its ability to improve evaluator beliefs through grade assignment. The school with greater first-stage investment has an advantage in the grading process. When grading is strategic, investment not only improves the likely ability of graduates, it also gives a school greater flexibility in designing its grading policy.

2 RELATED LITERATURE

A significant portion of the economics literature on grade inflation argues that inflation is a robust equilibrium phenomenon that often imposes a welfare cost on employers or other evaluators; none of these papers consider the interaction between grading policies and incentives to invest in education quality. Ostrovsky and Schwarz (2010) consider an assortative stable matching in a labor market. Vacancies are distinguished by desirability, and graduates are distinguished by their expected ability. These authors argue that in equilibrium, under certain circumstances, schools do not completely reveal the ability of their graduates to potential employers, assigning transcripts to students in a way that confounds employer beliefs about graduate ability. Popov and Bernhardt (forthcoming) consider a model of strategic grade assignment with a continuum of student abilities and two grades. They show that universities with better distributions of student abilities set lower grading standards; whereas a social planner would set a higher grading standard at a better university. Chan, Li and Suen (2007) consider a game in which schools know the distribution of its own students’ abilities, but an employer does not. They show that in equilibrium, schools will inflate grades by (sometimes) assigning a higher proportion of good grades than there are good students at the school. If they could do so, schools would benefit from a commitment to honest grading. Much of the grade inflation literature assumes (as we do) that evaluators can observe the grading policy utilized by a school prior to evaluating the graduates. Bar, Kadiyali and Zussman (2012) endogenizes the amount of information a school makes public about its grading policies. In their model, students strategically choose courses with different difficulties and different degrees of grade inflation in order to affect employer perceptions about their abilities. They then contrast the impact of disclosing grading information to students prior to course selection and to employers along with transcripts. They find that disclosure of grading policies to students affects course selection decisions, and that disclosing information to employers benefits students who elect to

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5 The model is designed in such a way that it is a school with a worse distribution of student abilities that sometimes inflates grades. This result somewhat contradicts empirical evidence and conventional wisdom, that grade inflation is more extreme at \textit{ex ante} better schools. We find that a better school inflates grades more than a lower quality school.

6 We discuss alternatives to this assumption in Section 5.
enter strictly graded courses, and hurts those who select lenient courses. Overall, this can damage aggregate student welfare. Like the rest of this literature, we also find that grade inflation is a robust equilibrium phenomenon. However, we reach different conclusions regarding the costs and benefits of grade inflation. We differ from the literature in that we allow schools to invest in the quality of education, and we find that schools invest more in the presence of grade inflation. Grade inflation leads to better-educated graduates and may increase the payoffs of the employers hiring these graduates.

Several authors consider the teacher-student relationship in the principal-agent framework. In this interaction, the teacher’s goal is to incentivize student effort through the design of grading policies. By considering exam performance as a game of status, Dubey and Geanakoplos (2010) demonstrate that in certain circumstances students are best motivated to exert effort when their exact exam performance is not revealed. Instead, it is more effective to publicly reveal performance information in broad categories (like letter grades). Related results are found in Zubrickas (2010) who shows that if the market (or other subsequent evaluator) cannot observe an individual teacher’s grading practices, the teacher responds by grading leniently. Our analysis complements this strand of the literature. In both Dubey and Geanakoplos (2010) and Zubrickas (2010) the teacher initially commits to a grading policy in order to incentivize subsequent student effort. Meanwhile, in our work, a school’s initial investment decision determines its subsequent advantage in the grading stage game. The interaction of student effort and grading policy is closely related to the interaction of school investment and grading policy. While we don’t explicitly address the issue of commitment to grading policy as a means of incentivizing student effort, we address the issue of student effort (observed or not) in Section 5.

Other authors consider issues to do with evaluation and certification but do not focus on the issue of grade inflation. Taylor and Yildirim (2011) consider the interaction of an evaluator’s performance standard and an agent’s unobservable effort. They find that the evaluator often benefits by committing to ignore information about agent attributes during the assessment process. In their framework, agent effort is endogenous, but the evaluator’s signal structure is exogenous; this is in contrast to our framework which includes both endogenous agent effort (i.e., school investment) and signal structures (grading policies), as well as competition between agents. In the context of industrial organization, Lizzeri (1999) studies the incentives of a rating agency to disclose information about product quality. A seller with private information about quality, has an opportunity to visit a rating agency with the capability to determine and certify quality. This rating agency commits to a disclosure rule (a stochastic mapping from qualities into reports) and to a price for this certification. The author shows that in a variety of important cases, the rating agency reveals a minimal amount of information to the market but appropriates a large share of the surplus.

3 A SIMPLE MODEL OF GRADE INFLATION AND INVESTMENT

Consider a three-stage game between two schools and an evaluator. First, each school invests in quality, which determines a graduate’s likely ability. Schools observe investment in school quality and then simultaneously choose grading policies, which determine how transcripts are assigned to graduates of different abilities. Finally, each school produces a single graduate. The evaluator observes school investments, graduate transcripts and grading policies, and awards a single prize to the graduate from one of the two schools.

\footnote{An important distinction between this paper and the rest of the literature, including our own paper, is that grading policies in different courses are (for most of the paper) exogenous. The key question is whether these policies are disclosed to different parties, but the bulk of the analysis is done with exogenous grading policies. They also consider endogenous grading policies, derived from faculty preferences for lenience or strictness.}

\footnote{See Moldovanu, Sela and Shi (2007).}
The prize could be a desirable job, admission to a prestigious law school or university, or an elite scholarship; the evaluator could be a recruiter, admissions officer, or representative of a scholastic trust. The evaluator prefers to assign the prize to a “high-ability” student. High-ability graduates are those who are likely to excel in the most-challenging post-graduation environments, whether they involve joining a prestigious company, or attending a top graduate school or college. Meanwhile, a school benefits whenever its graduate receives the prize, independent of his true ability.

In the first stage of the game, each school $i \in \{\alpha, \beta\}$ simultaneously chooses its quality, $q_i \in [0,1]$. School $i$’s graduate is “high-ability” ($\tau_i = h$) with probability $q_i$ and “low-ability” ($\tau_i = l$) with probability $1 - q_i$. Since evaluators want to select high-ability students and schools want their students selected, schools benefit from offering high education quality. However, improving school quality is costly in terms of resources or effort. To achieve quality $q_i$, school $i$ must pay a convex cost $C_i(q_i)$ where

$$C(q_i) = \frac{q_i^2}{\rho_i^2}.$$  

Parameter $\rho_i$ determines the marginal cost of quality, with higher values representing lower marginal costs. This parameter may represent the availability of resources for the schools, for example their infrastructures, endowments, or donor bases. Alternatively, it may represent characteristics of the student body (e.g. preparation or test scores). In this section, we focus on the case where $\rho_i = \rho_j = \rho$, and where $0 < \rho < \sqrt{2}$. Each school’s $q_i$ becomes public at the end of the first stage.

In the second stage of the game, the schools simultaneously select grading policies. When schools make this decision they know school qualities, but they do not know the realized ability of either graduate. In this section, we focus on a simple class of inflationary grading policies. A school can assign only two transcripts: a good transcript ($t_i = G$), or a bad transcript ($t_i = B$). Each school always assigns a good transcript to a high ability graduate, but may assign either a good or bad transcript to a low ability student. School $i$ chooses the probability $\theta_i \in [0,1]$ with which a low ability graduate draws a good transcript.

$$Pr(t_i = G|\tau_i = h) = 1 \text{ and } Pr(t_i = G|\tau_i = l) = \theta_i.$$  

If $\theta_i = 0$, then school $i$ always assigns good grades to high ability students and bad grades to low ability students. The greater the value of $\theta_i$, the higher the probability that a low ability graduates is assigned a good transcript. We refer to variable $\theta_i$ as the level of grade inflation at school $i$. In later sections of the paper, we offer a generalization in which schools design grading policies without restriction to a particular parametric class.

We can interpret the process of assigning grades as a process of disclosure, similar to the one in Lizzeri (1999). Following this interpretation, schools choose a grading policy (as described above) prior to learning the student’s ability. They then observe student ability, and generate a transcript for their student in accordance with their set grading policies. Alternatively, we may interpret a school’s grading policy as

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9Greenwald, Hedges and Laine (1996) perform a meta-analysis, demonstrating a substantial link between school resources and student achievement, while Hanushek (2006) provides evidence that increased resources at schools do not necessarily translate into better educational outcomes. However, even those finding little evidence of the link between spending and performance do not necessarily claim that additional resources could not be beneficial. They claim that this link may not be observed strongly in the data because schools do not allocate their resources in the most effective way possible. As described in a previous footnote, Rivkin, Hanushek and Kain (2005) demonstrate that teachers play an important role in educating students, and may be costly to hire and retain.

10In this interpretation, student’s true abilities are verifiable. In this way, an outsider, perhaps an accreditation agency, has the ability to monitor the school’s grade assignments to ensure that it adheres to its stated disclosure rule. While the evaluator would like to demand that the school release graduate’s true abilities in the second stage, we assume that a school’s choice of
a signal of student ability, designed by the school. Following this interpretation, schools do not directly observe the ability of their students; rather, they subject their students to a “test” of their own choosing. The verifiable outcome of the test provides a signal about student ability. Choosing the design of the test, the schools control the informativeness of the signals they produce about their students. In this way, grading can be viewed as a process of Bayesian persuasion, as described by Kamenica and Gentzkow (2011).

In the third stage of the game, each school’s graduate is evaluated and the evaluator awards the prize. She makes this decision after observing the quality of each school, the grading policy at each school, and each graduate’s realized transcript. If she awards the prize to a high-ability graduate, she receives payoff one. Otherwise, her payoff is zero. The evaluator’s expected payoff of offering the prize to each graduate is equal to the posterior probability that the graduate is high ability. It is therefore sequentially rational for her to offer the prize to the graduate whom she believes is more-likely high ability. If she holds the same beliefs about each graduate then she randomizes fairly between them. Once the prize is awarded, the true type of the recipient is revealed and payoffs are realized.

### 3.1 FULLY-REVEALING BENCHMARK

We begin by describing the outcome of the game for a benchmark in which grading policies are always fully informative about student quality, i.e., \( \theta_i = 0 \) for both schools. The fully-revealing benchmark may result from a rule put into place by university accreditation agencies, requiring that member institutions adhere to certain grading standards. It may also be a consequence of entrance or licensing exams given to all graduates applying for a position. Entrance exams for undergraduate and graduate study, as well as industry licenses and board certifications all provide evaluators with independent assessments about graduate ability. The more comprehensive the exam, the greater its potential to distinguish student abilities. In principle, such exams may be sufficiently informative to fully reveal student types, making transcripts irrelevant.

School \( i \)'s graduate receives the prize with probability \( 1/2 \) when the realized abilities of the graduates are the same, and with probability 1 when the graduate is high ability and his competitor is low ability. School \( i \)'s expected payoff given school quality investments \( (q_i, q_j) \) is

\[
u_i(q_i, q_j) = q_i(1-q_j) + \frac{1}{2}(q_iq_j + (1-q_i)(1-q_j)) - \frac{q_i^2}{\rho_i^2} = \frac{1}{2}(1+q_i-q_j) - \frac{q_i^2}{\rho_i^2}.
\]

As is evident from the above expression, the marginal benefit of improving quality for either school is independent of the other school’s investment. It is therefore not surprising that when schools choose investment in stage one, each school has a dominant strategy: \( q_i = \rho_i^2/4 \). The evaluator’s payoff is the probability that at least one graduate has high ability:

\[
u_e = 1 - (1-q_i)(1-q_j) = \frac{\rho_i^2 + \rho_j^2}{4} - \frac{\rho_i^2 \rho_j^2}{16}.
\]

When \( \rho_i = \rho_j = \rho \), these expressions simplify to \( q_i = \rho^2/4 \) and \( u_e = \rho^2/2 - \rho^4/16 \). In the next sections we compare educational investment and evaluator payoffs with strategic grading to this benchmark.

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11 In this section, we confine our analysis to a choice of signal structure in a simple parametric class. In a later section of the paper, we apply techniques from Kamenica and Gentzkow (2011) to study the strategic choice of grading policy without this restriction.

12 We assume that awarding the prize to a low-ability graduate dominates assigning the prize to no one. This assumption is not necessary for our results, but streamlines the exposition drastically. We also believe that this assumption is realistic in a variety of settings.
3.2 EQUILIBRIUM WITH GRADE INFLATION

We solve for Perfect Bayesian Equilibria of the three stage game in which schools strategically invest in quality and then choose inflationary grading policies. We proceed by backwards induction.

3.2.1 Stage Two: Strategic Grade Inflation

We characterize equilibrium strategies and payoffs in the grading stage of the game, following investment decisions \((q_i, q_j)\) in stage one.

If school \(i\) chooses investment level \(q_i\) in stage one and grade inflation level \(\theta_i\) in stage two, then the probability that a graduate receives a good transcript is

\[
r_i = q_i + \theta_i(1 - q_i).
\]

This probability is increasing in the school’s level of grade inflation. At the same time, the evaluator’s updated belief that a graduate with a good transcript is truly high-ability is derived from Bayes’ rule:

\[
g_i = \frac{q_i}{q_i + \theta_i(1 - q_i)} = \frac{q_i}{r_i}
\]

A school’s choice of grade inflation \(\theta_i\) involves a simple tradeoff. Increasing \(\theta_i\) assigns the good transcript to low ability students more-often. This increases the overall probability that the evaluator observes a good transcript, but at the same time causes the evaluator to view good transcripts with greater skepticism. Thus, the decision to inflate grades involves a tradeoff between the probability a graduate “looks good” and how good such a graduate looks.

If \(\theta_i\) and \(\theta_j\) are such that \(g_i > g_j\), then the evaluator will award the prize to graduate \(i\) whenever both graduates have good transcripts. In this case, therefore, graduate \(i\) receives the prize whenever he realizes a good transcript, regardless of the other student’s transcript realization, and will also receive the prize with probability \(1/2\) when both students realize a bad transcript. Therefore, following stage one investment decisions \((q_i, q_j)\), school \(i\)’s expected payoff as a function of the inflation levels \((\theta_i, \theta_j)\) is as follows:

\[
u_i(\theta_i, \theta_j|q_i, q_j) = \begin{cases} 
 r_i(1 - r_j) + \frac{1}{2}(1 - r_i)(1 - r_j) & \text{if } g_i < g_j \\
 r_i + \frac{1}{2}(1 - r_i)(1 - r_j) & \text{if } g_i > g_j \\
 r_i(1 - \frac{1}{2}r_j) + \frac{1}{2}(1 - r_i)(1 - r_j) & \text{if } g_i = g_j.
\end{cases}
\]

We make two observations about best responses. First, given \(g_j\), school \(i\) always prefers to select \(\theta_i\) for which \(g_i\) is marginally higher than \(g_j\), rather than for which \(g_i = g_j\). This means that an equilibrium involving \(g_i = g_j\) is possible only at \(g_i = g_j = 1\) (which corresponds to the no inflation case of \(\theta_i = \theta_j = 0\)) at which point it is no longer possible to increase \(g_i\). Second, notice that among all grading policies for which \(g_i < g_j\), school \(i\) prefers \(g_i = g_i\) (which corresponds to the uninformative grading policy \(\theta_i = 1\)).

Taken together, these observations suggest that the equilibrium of the grading stage is either a fully-revealing pure strategy equilibrium \(\theta_i = \theta_j = 0\), or a mixed strategy equilibrium.\(^{13}\)

\(^{13}\)If \(g_i < g_i\), then graduate \(i\) will be selected for certain if and only if his transcript realization is good and \(j\)’s transcript is bad; otherwise graduate \(i\) will either not be chosen, or the evaluator will randomize. By choosing an uninformative grading policy, school \(i\) assures that its graduate’s transcript will be good, and therefore guarantees that its graduate will be selected if the other school’s graduate’s transcript is bad. Therefore, if an asymmetric pure strategy equilibrium exists, then one school must play an uninformative strategy. However, because of an open set issue, the best response to an uninformative grading policy is undefined.

\(^{14}\)The mixed strategy equilibrium requires that schools select grading policies without perfectly anticipating the other school’s
Below, we describe the structure of the equilibrium of the second stage sub-game for any combination of initial investment in school quality, \( q_a \geq q_b \).

**Case I:** \( q_b \leq \frac{1}{2}(1 - q_a) \). The lower quality school chooses \( \theta_b \) according to a mixed strategy. The support for \( \theta_b \) is an interval contained within \((0, 1)\). Thus, the grading policy at the lower-quality school always inflates the grades of some—but not all—of its low-ability students. The higher quality school also chooses \( \theta_a \) according to a mixed strategy. Grade inflation is more extreme at the higher-quality school: its mixed strategy over levels of inflation first-order-stochastic dominates the mixed strategy of the lower quality school. Furthermore, the higher-quality school’s mixed strategy has a mass point on \( \theta_a = 1 \), the completely inflated (uninformative) grading policy.

**Case II:** \( \frac{1}{2}(1 - q_a) \leq q_b \leq 1 - q_a \). Here, the lower-quality school chooses a mixed strategy with a mass point at \( \theta_b = 0 \), and support over an interval inside \((0, 1)\). The higher-quality school’s mixed strategy has mass points on both \( \theta_a = 0 \) and \( \theta_a = 1 \), and support over an interval inside \((0, 1)\). As in Case I, the high quality school’s mixed strategy first order stochastic dominates the low quality school’s mixed strategy. The primary qualitative difference in the grading policies here as opposed to in Case I is that each school sometimes implements a fully-revealing grading policy.

**Case III:** \( q_b \geq 1 - q_a \). In this case, neither school inflates grades. Both schools choose fully revealing grading policies, \( \theta_a = \theta_b = 0 \).

Equilibrium grading policies at both schools often involve grade inflation. Whenever \( q_b < 1 - q_a \) each school utilizes an inflationary grading policy with positive probability; if \( q_b \leq \frac{1}{2}(1 - q_a) \) each school utilizes an inflationary grading policy with probability one. Furthermore, whenever grade inflation may take place in equilibrium \((q_b < 1 - q_a)\) the high-quality school inflates grades more-aggressively, as its mixed strategy over inflation levels first order stochastic dominates the lower quality school’s mixed strategy.

### 3.3 Stage One: Investment

In order to analyze the period one school investment choices, we present the expected payoff of each school in this equilibrium. In Case III \((q_b \geq 1 - q_a)\) each school uses a fully revealing grading policy, which gives school \( i \) payoff:

\[
u_i(q_i, q_j) = \frac{1}{2}(1 + q_i - q_j) - \frac{q_i^2}{\rho^2}\]

In the Appendix, we show that the payoff functions for Cases I and II \((q_b \leq 1 - q_a)\) are

\[
u_a(q_a, q_b) = \frac{q_a}{q_a + q_b} \quad \text{and} \quad \nu_b(q_a, q_b) = \frac{q_b}{q_a + q_b}.
\]

Although it is not the focus of this paper, it is remarkable to note that our model provides a micro-foundation for the often-used contest success functions proposed by Tullock (1980).

At the investment stage, anticipating the stage two equilibrium derived above, the payoff function for choices, and that adjusting one’s grading policy is infeasible after the first stage. The mixed strategy equilibrium only arises in the simple version of the game. In the generalized version of the model considered later, schools do not randomize between grading policies, although any given grading policy generates transcripts stochastically.
school $i$ is

$$u_i(q_i, q_j) = \begin{cases} 
\frac{q_i - \frac{q^2}{p^2}}{q_i + q_j} & \text{if } q_i \leq 1 - q_j \\
\frac{1}{2}(1 + q_i - q_j) - \frac{q^2}{p^2} & \text{if } q_i > 1 - q_j
\end{cases}$$

This payoff function is continuous everywhere and is differentiable everywhere except for possibly $q_i = q_j$. The non-differentiability arises when the investment levels change the equilibrium from Case II to Case III. Despite this non-differentiability, the equilibrium investment levels are simple to characterize. Given that $\rho \leq \sqrt{2}$, the symmetric equilibrium investment levels are $q_i = \sqrt{2}ho$. Investment is higher than in the fully-revealing benchmark, and grade inflation takes place with non-zero probability on the equilibrium path.\(^{15}\)

The freedom to inflate grades spurs schools to invest more to educate graduates.\(^{16}\) Increased investment has a direct effect on the ability of a school’s graduate: by investing more, a school increases the expected ability of its graduate. With noisy grading, increased investment not only increases the probability that a school’s graduate is high ability, for any grading policy increased investment also causes the evaluator to have higher expectations about a graduate who earns a high grade. This increased optimism on the part of the evaluator allows the school to inflate grades more aggressively in the second stage without compromising the evaluator’s assessment of its graduate, giving the school an advantage in the grading stage. By investing more a school gains additional “leeway” in the grading stage, an effect absent when grades must fully distinguish types. Thus increased investment is more valuable when grading is noisy, as often arises in the equilibrium with grade inflation. Although it seems intuitive that schools would treat the ability to inflate grades as a substitute for educational investment, our analysis suggests that these are actually complementary.

When transcripts are inflated, the evaluator has less information about realized graduate ability than in the fully-revealing benchmark, and therefore she may incorrectly assign the prize. At the same time, education quality is higher: all graduates are more-likely to be high ability. Thus a tradeoff exists between incentives for schools to invest ex ante in developing student ability, and the evaluator’s ability to select high ability graduates ex post. In the simple model of this section one of these effects always dominates the other, as shown in the following proposition.

**Proposition 3.1** The equilibrium with grade inflation has a higher level of investment and a higher evaluator payoff than the fully-revealing benchmark.

Allowing grade inflation increases investment in school quality, evaluator payoffs and aggregate graduate ability. Eliminating grade inflation can decrease school quality, evaluator payoffs and aggregate graduate ability. In the next sections we explore the interaction between grading policy and education investment in an extended version of the model.

## 4 General Grading Policies

In this section we analyze a version of the model in which we allow schools to freely choose their grading policies, without restricting their choices to policies that only inflate the grades of low ability students. We also incorporate asymmetric costs of education.

As in the simple version of the model, schools invest in education in stage one. Each school’s education level $q_i$ determines the probability that its graduate is high ability, but costs the school $C_i(q_i) = \frac{q^2}{\rho_i^2}$. In this

\(^{15}\) If $\rho > \sqrt{2}$ then the symmetric equilibrium investment levels are $q_i = \sqrt{2}$, as in the fully-revealing benchmark. No grade inflation takes place on the equilibrium path. In this section, we restrict attention to the case of lower $\rho$.

\(^{16}\) Note, $\rho < \sqrt{2} \Rightarrow \sqrt{2} \rho > \frac{\sqrt{2}}{\rho}$. 

section we allow for asymmetries in the parameter \( \rho_i \), reflecting differences in a school’s access to resources, students’ background preparation, or other factors influencing the school’s cost of increasing student ability. School \( \alpha \) has access to better resources than school \( \beta, 0 < \rho_\beta < \rho_\alpha < 2 \).

In this version of the game, we allow a school to employ any system of grading it would like, without restricting it to a particular parametric class (e.g., only two possible transcript realizations, and a restriction that all high ability graduates receive the higher of the two grades). A grading policy at school \( i \) is represented by a pair of conditional random variables \((H_i, L_i)\). The transcript of a high-ability student is an independent realization of \( H_i \) and the transcript of a low-ability student is an independent realization of \( L_i \). For technical reasons, we focus on random variables \((H_i, L_i)\) for which the cumulative distribution function has a finite number of discontinuities or mass points. Except at mass points, \( H_i \) and \( L_i \) admit differentiable densities with support over an interval. We refer to random variables with this structure as valid. Any pair of valid random variables that satisfies the monotone likelihood ratio property is an admissible grading policy.\(^{17}\) If an evaluator observes a transcript which is in the support of \( H_i \) but not in the support of \( L_i \), then the evaluator can correctly infer that the graduate receiving that transcript is high ability. Similarly, if an evaluator observes a transcript in the support of \( L_i \) but not in the support of \( H_i \), then it infers that the graduate must be low ability. If the evaluator observes a transcript which is in the support of both \( H_i \) and \( L_i \), then some uncertainty remains about whether the graduate is high or low ability. Given the priors, grading policies, and transcript realizations, the evaluator’s posterior belief about ability is determined by Bayes’ rule.

This representation of a school’s grading policy is quite general, and includes any possible system of grading that utilizes a finite number of letter grades; it also allows for more complex grading schemes such as assigning students a numerical value in the interval \([0,100]\) or \([0,4]\). Frequently, actual transcripts are not limited to a numeric or letter score, as they typically provide a list of classes taken by semester, grades by class, and overall grade point average. Some schools also include class/grade distribution on transcripts (see Bar, Kadiyali and Zussman (2012)). This is perfectly consistent with our model, with the random variables \( H_i \) and \( L_i \) together with the prior determining the likelihood that each possible transcript is owned by a high-ability graduate. At the same time, because the underlying set of “nominal” transcript realizations is not specified (and grade inflation is by definition a nominal grading phenomenon), the general version of the model is unable to distinguish between grade inflation and manipulation of the grade distribution more generally. Any grading policy that takes place in equilibrium is perfectly consistent with a story of pure grade inflation, but is also consistent with countless other strategically equivalent grading schemes that result in the same information conveyed to the evaluator.\(^{18}\)

\(^{17}\)In this context the monotone likelihood ratio implies that transcripts are ordered in such a way that a greater transcript realization is associated with a greater posterior belief that the graduate is high ability; that is, higher transcripts always brings “good news” about graduate ability.

\(^{18}\)To see this, let us jump ahead briefly to the equilibrium grading scheme described in Lemma 4.2. The lemma describes a grading policy in which some transcript realizations perfectly reveal that a graduate is low quality, some transcripts perfectly reveal that a graduate is high quality, and other transcripts leave an evaluator uncertain about the graduate type. The type of noise that arises in equilibrium may be achieved through grade inflation. Suppose that with perfectly informative grading policies, schools always award transcripts full of Ds and Cs to low ability students and transcripts full of Bs and As to high ability students. Such transcripts fully distinguish the two student types. With grade inflation, the schools may start to award some of the low ability students transcripts with Bs and As which were originally reserved for the high ability students. At the same time, the schools may continue to reserve some of the high ability transcripts (e.g., straight As or A+s) for only high ability students. Such a process of grade inflation can approximate the equilibrium grading structure. The grading process could also be captured if schools deflated grades of some high ability students; but the deflation alternative is inconsistent with the empirical observations of school grading policies. We therefore focus our discussion on grade inflation.
4.1 STAGE TWO: GRADING POLICIES

Following the approach of Bayesian Persuasion developed by Kamenica and Gentzkow (2011), we first describe a representation of grading policies that considerably simplifies the analysis of the second stage of the game. The evaluator’s behavior is determined by her posterior belief about graduate ability. For a school, the only payoff-relevant aspect of a grading policy is the probability distribution over the evaluator’s posterior belief generated by the grading policy. We therefore represent any feasible grading policy \((H_i, L_i)\) by a single random variable \(\Gamma_i\) from which the evaluator’s posterior belief about a graduate’s ability is drawn. To elaborate, suppose that given the prior belief about student ability, transcript realization \(x\) from grading policy \((H_i, L_i)\) results in the evaluator having posterior belief \(\gamma = Pr(t = h|x)\). Along with the prior, grading policy \((H_i, L_i)\) also determines the probability distribution of \(x\), which is itself a random variable \(X\). Thus the prior belief and grading policy determine the \(ex\ ante\) distribution of the evaluator’s posterior belief, \(\Gamma_i = Pr(t = h|X)\), and this random variable captures all payoff-relevant aspects of the underlying grading policy. Random variable \(\Gamma_i\) is valid, has support confined to the unit interval, and according to the Law of Total Expectation has expectation equal to the prior, \(q_i\). In the next lemma (adapted from Kamenica and Gentzkow (2011)), we show that these are the only substantive restrictions on the \(ex\ ante\) posterior beliefs that can be generated by a grading policy.

**Lemma 4.1 (Bayesian Persuasion representation)** Consider any valid random variable \(\Gamma_i\) with support confined to the unit interval and expectation \(q_i\). If the prior belief that a student is high ability is \(q_i\), then there exists a grading policy \((H_i, L_i)\) for which the \(ex\ ante\) posterior belief is \(\Gamma_i\).

This lemma considerably simplifies the analysis. All payoff-relevant aspects of a grading policy are summarized by a single random variable, representing the \(ex\ ante\) distribution of the evaluator’s posterior belief. The lemma shows that any random variable with support inside \([0,1]\) and mean equal to the prior is the \(ex\ ante\) posterior belief for some grading policy.\(^{19}\) The analysis can therefore focus on an alternative version of our original game in which each school chooses \(\Gamma_i\) rather than \((H_i, L_i)\), as long as \(\Gamma_i\) is valid, has support in the unit interval, and expectation \(q_i\). We refer to the choice of \(\Gamma_i\) as a choice of a grading policy; although \(\Gamma_i\) technically represents an entire payoff-equivalent class of grading policies.

Schools benefit when their graduates generate high realized values of the posterior belief. If possible, a school would like \(\Gamma_i\) to only result in high realizations of the posterior belief. However, the school is constrained by the expected ability of its student, determined by its investment in education, \(q_i\). Because the evaluator rationally updates her beliefs about student ability, the Law of Iterated Expectation requires that the expected value of the posterior belief generated by any feasible grading policy must be equal to the prior probability \(q_i\). The freedom to choose a grading policy allows a school to strategically reveal or conceal information about the ability of its graduates, but it can not use its grading policy to make its graduates appear to be better, on average, than they truly are.

In the third stage of the game, each school produces one graduate. The grading process generates realizations \((\gamma_\alpha, \gamma_\beta)\) of the posterior belief random variables \((\Gamma_\alpha, \Gamma_\beta)\). The evaluator then awards the prize to the graduate she believes is most qualified, selecting the graduate with the higher realized \(\gamma_i\). When \(\gamma_\alpha = \gamma_\beta\), both graduates are selected with equal probability. The expected payoff to school \(i\) when schools choose grading policies \((\Gamma_i, \Gamma_j)\) is therefore \(Pr(\Gamma_i > \Gamma_j) + \frac{1}{2}Pr(\Gamma_i = \Gamma_j)\).

With this representation of grading policies, the evaluation stage of the game is closely related to the full-information all-pay auction, with \(\gamma_i\) corresponding to the bid of player \(i\) and \(\Gamma_i\) to the player’s mixed

\(^{19}\)In fact, large class of grading policies.
strategy. The fundamental difference between the evaluation stage of our game and a full-information all-pay auction is that the “bid distribution” in our environment must maintain an expected value equal to \( q_i \) (the average ability of a graduate). This feature of our analysis is shared by Wagman and Conitzer (2011), who consider an all pay auction with a constraint on the mean bid. Our analysis requires an addition constraint that the random variables (designed by schools) are limited to realizations on the unit interval, as the realizations represent probabilities. While the framework representing the grading stage is similar to the all pay auction considered by these authors (after the Bayesian Persuasion representation is applied), the focus of our analysis is on the interaction between incentives to invest, and a school’s freedom to design grading policies, a consideration not included in the other analysis.

Suppose one of the schools, which we refer to as \( A \), invests more in school quality than the other school, which we refer to as \( B \). That is, \( q_b \leq q_a \). Because school \( A \) has invested more, we refer to \( A \) as the high-quality school.

The following Lemma characterizes the Nash equilibrium of the second stage game, for each possible combination \((q_a, q_b)\).

**Lemma 4.2 (Strategic Grading Equilibrium)** The unique Nash equilibrium of the grading stage is given by the following combination of grading policies \((\Gamma_a, \Gamma_b)\):

- When \( q_a \leq \frac{1}{2} \):

  \[
  \Gamma_a = U[0, 2q_a], \quad \Gamma_b = \begin{cases} 
  0 & \text{with prob } \frac{1}{q_a} \\
  U[0, 2q_a] & \text{with prob } \frac{q_b}{q_a}.
  \end{cases}
  \]

- When \( q_a > \frac{1}{2} \):

  \[
  \Gamma_a = \begin{cases} 
  U[0, 2(1-q_a)] & \text{with prob } \frac{1}{q_a}, \\
  1 & \text{with prob } \frac{1}{q_a} - 1
  \end{cases}, \quad \Gamma_b = \begin{cases} 
  0 & \text{with prob } \frac{1}{q_a} \\
  U[0, 2(1-q_a)] & \text{with prob } \frac{q_b}{q_a} (\frac{1}{q_a} - 1)
  \end{cases}
  \]

When \( q_a \leq \frac{1}{2} \), both schools are more-likely to produce a low-ability graduate than a high-ability graduate. In this situation, the school with the quality advantage chooses a grading policy that leaves the evaluator less-than-fully informed about the quality of its graduate, employing a grading policy that generates a uniform distribution over posterior beliefs centered on the prior. Thus every transcript that the high-quality school assigns can be assigned to both a high ability or low ability graduate. By always maintaining a mix of both types for any given transcript, the school utilizes its (unlikely) high ability graduate to improve the perception of its (likely) low ability graduate. In equilibrium, the school \( B \) adopts a grading policy that mimics the grading policy of the school \( A \) with one exception: in order to maintain \( \mathbb{E}[\Gamma_b] = q_b \), school \( B \) sometimes issues a transcript that reveals (for certain) that its graduate is low ability. The low-quality school responds to this by sometimes issuing a low-ability student a transcript that fully reveals his type, but otherwise utilizing the same grading policy as the high quality school.\(^{22}\)

\(^{20}\)Although Wagman and Conitzer (2011) do consider an initial investment stage, their two stage game in outcome equivalent to a standard all pay auction given the assumption of linear costs.

\(^{21}\)As one would expect, when we derive the equilibrium of the investment stage, the advantaged school \( A \) will choose to invest more in education, and will play the role of school \( A \) on the equilibrium path.

\(^{22}\)In the appendix we argue that this mimicry exists not only on the level of the posterior distribution, but also exists on the underlying grading policy. In order to achieve the equilibrium posterior belief distribution school \( B \) can use a grading policy
When \( q_a > \frac{1}{2} \), the higher-quality school is more likely to produce a high-ability graduate than a low-ability graduate, so that high ability graduates are relatively likely. In this case, the higher-quality school does not need to assign every transcript to a mix of both high and low ability graduate. Like the previous case, school A never reveals that its graduate is low ability, but unlike that case, it sometimes reveals that its graduate is high-ability. To do this, the school reserves some “outstanding” transcripts (i.e., 4.0 GPA, Honors Program, etc.) for only high ability graduates, and all other transcripts are assigned to a mix of both high- and low-ability graduates. The lower-quality school again responds by sometimes revealing when its graduate is low ability, but otherwise mimicking the posterior belief distribution (and underlying grading policy) of the other school.

Both schools’ grading policies exhibit compression at the top of the transcript distribution. It is argued (see Cizek 1996) that grade compression is a natural consequence of grade inflation. Because no grade higher than an A exists, as schools assign higher grades to their graduates, “As remain As, but Bs become As, Cs become Bs, and so on. The result is that it takes less to achieve an A.” Thus, when schools inflate their grades, the inference that the evaluator draws from good transcripts becomes worse, but low grades still convey the same negative information they did in the absence of grade inflation (although they are less-likely to be assigned). The equilibrium we find conforms to this pattern. In each equilibrium of the game, some transcripts convey to evaluators that the student is low-ability with probability one. The school with a lower initial investment assigns such transcripts with positive probability. The high-investment school does not assign such transcripts with positive probability, avoiding distinguishing any of its graduates as being low ability for sure. Thus in all equilibria the worst grade still conveys that the graduate is low ability, but it is assigned less often by both schools than under fully-revealing grades.

At the top of the distribution, things are more subtle. In the first type of equilibrium, the best belief in the support of each school’s posterior belief distribution is less than one. This means that the best transcript possible might be assigned to the low ability student.\(^{23}\) In the second type of equilibrium, both schools reserve the best-possible transcripts for high ability students. Outstanding transcripts are therefore “uncompressed.” However, the inference that an evaluator draws from anything but an outstanding transcript becomes worse.

The main properties of the second stage equilibrium are summarized in the following proposition.

**Proposition 4.3** If school A invests more than school B (i.e., \( q_b \leq q_a \)), then in equilibrium of the second stage:

- No equilibrium exists in which either school uses a fully-revealing grading policy.
- School A does not reserve any transcripts realizations for low-ability students. All transcripts assigned to low ability students could also be assigned to high-ability students.
- School B reserves certain transcript realizations for low-ability students, but otherwise uses the grading policy as school A.
- The grading policy at school B is more Blackwell informative than the grading policy at school A.

Consistent with previous literature on grade assignment, we find that fully-revealing grades are not part of an equilibrium. This means that the fully-revealing equilibrium found in section 3 is the consequence of \((H_b, L_b)\) that is identical to school A’s, \((H_a, L_a)\) except that \(L_b\) contains a special realization (or set of realizations) that is issued only to low ability students.\(^{23}\) Because even the best transcript is in the support of \(L_i\), the evaluator can not be sure that a student with the best possible transcript is really high ability.
the binary grade structure assumed in that section. Furthermore, consistent with previous literature and conventional wisdom, we find that schools whose students are less-likely to be high ability use grading policies that are more informative.

In the next section, we consider the interaction between grading policy and a school’s initial investment in student ability. In order to facilitate this analysis, we summarize and discuss the equilibrium payoffs for the schools:

\[ u_a(q_a, q_b) = 1 - \frac{q_b}{2q_a} \quad \text{and} \quad u_b(q_a, q_b) = \frac{q_b}{2q_a}. \]

As expected, school A has a payoff advantage in equilibrium. The structure of the equilibrium reveals the source of this advantage. If school B were able to perfectly mimic school A’s posterior belief distribution, it could completely neutralize school A’s initial advantage, guaranteeing a payoff of \( \frac{1}{2} \) for both schools. However, because the prior belief about ability is lower at school B, school A’s posterior belief distribution is not feasible. In order to stay competitive, school B sometimes reveals that its graduate is low ability, which allows B to plausibly mimic A’s posterior belief distribution the rest of the time. Conditional on B not revealing its graduate to be bad, each school acts in an identical way, and therefore both schools expect the same payoff. However, if school B reveals that its graduate is bad, graduate A is assigned the prize for certain. This happens with probability \( 1 - \frac{q_b}{q_a} \). Therefore, the greater the investment gap between schools, the more often B is forced to reveal that its graduate is low ability. School B has an incentive to close this gap in order to keep from having to reveal its graduate to be bad; meanwhile school A has an incentive to widen the gap, forcing school B to reveal that its graduate is low ability more often.

4.2 STAGE ONE: INVESTMENT IN SCHOOL QUALITY

In the first stage of the game, schools simultaneously invest in education quality, which affects the expected ability of their graduate. Remember, the \( \alpha \) school has a lower marginal cost of improving education quality compared to the \( \beta \) school; that is, \( 0 < \rho_\beta \leq \rho_\alpha \).

Anticipating the second period equilibrium grading policies, in the first stage of the game each school \( i \in \{\alpha, \beta\} \) expects payoff \( u_i(q_a, q_\beta) \) from quality investments \( q_a \) and \( q_\beta \), where

\[
  u_i(q_a, q_\beta) = \begin{cases} 
    \frac{q_a}{2q_i} - \frac{q_\beta^2}{r_i^2} & \text{if } q_i \leq q_j \\
    1 - \frac{q_a}{2q_i} - \frac{q_\beta^2}{r_i^2} & \text{if } q_i > q_j.
  \end{cases}
\]

This function is differentiable, continuous, and concave in \( q_i \). The schools simultaneously choose \( q_i \) and \( q_\beta \) to maximize their expected payoff. In equilibrium, they choose

\[ q_\alpha = \frac{\sqrt{\rho_\alpha \rho_\beta}}{2} \quad \text{and} \quad q_\beta = \frac{\rho_\beta^2}{2 \sqrt{\rho_\alpha \rho_\beta}}. \]

If \( \rho_\beta \) increases, then the difference between \( \rho_\alpha \) and \( \rho_\beta \) decreases, the competition between the two schools becomes more intense, and the schools both respond by increasing investment in education quality. A decrease in \( \rho_\alpha \) has a similar effect, causing the disadvantaged school to invest more in education quality. However, the increase in the costs of investment for the advantaged school more than offsets the increased competitive pressure, and the advantaged school decreases investment in education quality as \( \rho_\alpha \) decreases.

\[ 24 \text{This effect can be clearly seen by decomposing the payoff functions for the schools in the following way: } u_a(q_a, q_b) = \frac{2q_b}{q_a} \left( \frac{1}{2} \right) + (1 - \frac{4q_b}{q_a}) \quad \text{and} \quad u_b(q_a, q_b) = \frac{4q_b}{q_a} \left( \frac{1}{2} \right). \]
4.3 STRATEGIC GRADING VERSUS FULLY-REVEALING GRADING

In this section, we compare outcomes in the game when schools strategically choose grading policies to the outcomes in the fully-revealing benchmark (for which the equilibrium is derived in Section 3.1). If school quality is exogenous (a standard assumption in the literature), then requiring fully-revealing grades makes the evaluator better off compared to strategic grading. With exogenous quality, the evaluator’s payoff depends only on her ability to select high-ability graduates. Because strategic grading is noisy, the evaluator would benefit by requiring that grades completely reveal graduate ability.

We compare outcomes when school quality is endogenous. Here, we find that allowing strategic grading brings surprising benefits: as in the simple game presented previously, when schools grade strategically they often invest more to improve education quality. This benefit also may reverse the evaluator’s welfare ranking; with endogenous investment the evaluator prefer strategic grading to fully-revealing grading if the school’s investment costs are not too different. Even if she had the capability to eliminate all noise from student transcripts, it may not be in her interest to do so.

Our first main result concerns school investment:

**Proposition 4.4** (Fully-revealing versus strategic grading) Requiring schools to use fully-revealing grading policies rather than strategic grading

- always decreases investment in school quality by the disadvantaged school
- decreases investment in school quality by the advantaged school if and only if $\rho_\beta \in (\frac{1}{4} \rho_\alpha, \rho_\alpha)$,
- decreases average school quality (and expected aggregate student ability) if and only if $\rho_\beta \in (\bar{r}, \rho_\alpha)$, where $\bar{r} \leq \frac{1}{4} \rho_\alpha$.

Schools often invest more in quality when they grade strategically. To develop intuition for this result, recall that with fully-revealing grading the marginal benefit of increased investment for either school is fixed and equal to $\frac{1}{2}$. With strategic grading, however, the marginal benefit of investment is not fixed at $\frac{1}{2}$ for either school. As discussed previously, by closing the investment gap, the lower quality school, $B$, reduces the probability of revealing that its graduate is low ability, contesting the prize more-often. In equilibrium the prize is contested with probability equal to $\frac{q_a}{q_b}$. If it does not reveal a low ability graduate, school $B$ mimics the grading policy of school $A$, giving it an expected payoff of $\frac{1}{2}$ in this circumstance. Thus for school $B$ the marginal benefit of increased investment in quality is $\frac{1}{2q_a}$. This marginal benefit of investment is fixed (because the probability of contesting the allocation of the prize is linear in $q_b$), and is greater than the marginal benefit of investment with fully-revealing grades. The disadvantaged school, $\beta$, therefore always invests more with unrestricted strategic grading than with fully-revealing grading, in order to avoid conceding the prize to $\alpha$ as often. Conversely, school $A$ benefits from widening the investment gap by forcing school $B$ to concede the prize more-often in equilibrium. However, the probability that school $B$ concedes the prize, $1 - \frac{q_a}{q_b}$ depends on $q_a$ in a concave way. The marginal benefit of increasing quality for school $A$, $\frac{q_a}{2q_a}$ is diminishing. Thus, depending on the marginal costs of increasing quality, school $\alpha$ may choose either higher or lower investment when grading is unrestricted. If the schools are similar *ex ante*, then competition is most-fierce, and both schools invest more in education quality.

25As discussed in Section 3.1, marginally increasing investment results causes fewer ties in which both schools produce low-ability graduates and allows the investing school to win more often, and also causes more ties in which both schools produce high ability graduates, creating ties in situations in which the school would have lost for certain.
For fixed levels of school quality, the evaluator is worse off when schools choose strategic rather than fully-informative grading policies since she is less-able to determine a graduate’s ability from observing his transcript. When schools invest in quality, however, strategic grading may motivate schools to invest more in education quality, increasing the probability that graduates are high-ability. Although strategic grading makes the evaluator less able to determine each graduate’s ability, because of the increased investment, the evaluator may, overall, be better off. It is a simple matter to check that whenever \( \rho_\alpha = \rho_\beta \) the evaluator prefers strategic grading to the fully-revealing benchmark. By continuity, a region exists around the diagonal in which this result holds. This brings us to our second main result:

**Proposition 4.5** Requiring schools to assign fully-revealing grades rather than engage in strategic grading hurts the evaluator whenever the initial asymmetry between schools is not too large. For each value of \( \rho_\alpha \) there exists a value \( \bar{\rho} < \rho_\alpha \) such that, if \( \rho_\beta \in (\bar{\rho}, \rho_\alpha) \), then the evaluator equilibrium payoff is lower when grading policies are required to be fully-revealing.

Allowing schools the freedom to grade strategically can benefit the evaluator. Eliminating grade inflation can decrease both the average ability of graduating students, and the probability that the evaluator selects a high ability graduate.

### 4.4 TO GRADE OR NOT TO GRADE...

In this section we compare the evaluator’s payoff in the strategic, fully-revealing and uninformative grading equilibria. To do this, we first briefly describe the outcome of the game when grading policies provide evaluators with no information about graduate ability.

In equilibrium of the grading game, schools strategically choose grading policies which are neither fully-revealing nor uninformative. The previous analysis shows that because it motivates greater investment in education quality, strategic grading may lead to better outcomes than fully-revealing grading, improving aggregate student ability and evaluator welfare. Although the evaluator is best able to select high ability graduates with fully-informative transcripts, education quality is lower. It is natural to also consider a benchmark in which grades are uninformative. This benchmark applies if the evaluator bans (or commits to ignore) transcripts. It also applies to a situation of rampant grade inflation, in which transcripts convey no information about graduate ability. With uninformative grades, the evaluator’s ability to select the best graduate \textit{ex post} is undermined, as she has no information about realized ability. At the same time, the evaluator’s decision is based solely on a school’s investment level, which can provide the strongest incentives for investment.

We briefly describe the uninformative grading equilibrium. In the absence of grades, the evaluator assigns the job to the graduate who is more-likely to be high-ability given her prior belief. Thus, the evaluator assigns the job to the graduate of the school that invests more in education. The game between schools is therefore a full-information all-pay auction with asymmetric convex costs. In the absence of grades, competition over investment in education quality is especially significant, particularly when schools are evenly matched (that is, have similar investment costs). Fierce competition can lead to the highest expected ability of graduates. Although the evaluator cannot observe any information about realized graduate ability, the fierce quality competition between schools causes them to produce high-ability graduates more often, frequently giving the evaluator a higher expected payoff. We formally solve for the equilibrium in the appendix.

We demonstrate that a region of the parameter space exists in which each of the three delivers a higher payoff to the evaluator than the other two. The following proposition and accompanying figure summarize
the results of the three-way comparison.

**Proposition 4.6 (Three-Way Comparison)** A set of parameters exists for which each of the three alternatives, strategic, fully-revealing, and uninformative grading, delivers the evaluator a higher payoff than the other two.

In the figure above, whenever \((\rho_a, \rho_b)\) belong to Region I, the evaluator payoff is highest with uninformative grading; in Region II the evaluator payoff is highest with strategic grading, and in Region III the evaluator payoff is highest with fully-revealing grading.

The above figure is important for our analysis. It shows that the tradeoff between *ex post* selection and *ex ante* investment may be resolved with extreme policies (fully-informative or uninformative grading), but circumstances exist in which strategic grading (an intermediate case) is preferred to either of these extremes. This figure also suggests situations in which we might expect an evaluator to overweight school reputation, a proxy for quality or investment. By considering only investment, the evaluator effectively implements the uninformative benchmark. She could benefit from this whenever the costs of investment are similar. If the pool of schools that feed their graduates to a certain evaluator are similar, the evaluator could benefit by only considering investment. If school asymmetries are moderate, the evaluator benefits by allowing schools to grade strategically; if these differences are substantial, then fully-revealing grading is preferred.

5 \ ON THE MODEL’S ASSUMPTIONS

In this section we discuss some of the assumptions of the model in more detail.

5.1 \ COMPETITIVE STRUCTURE

Our model includes a single evaluator assigning a single prize, and one student per school. The framework incorporates the simplest assortative matching of graduates to prizes, allowing us to maintain tractability as we generalize other aspects of the analysis. While the interaction of matching and grade inflation is the primary focus of other papers in the literature, our focus is different. We simplify the matching aspect of the problem, in order to focus on the interaction of grading and investment.
If we consider random matching between graduates of each school and evaluators, then our results apply directly to situations with any number of graduates and evaluators.\textsuperscript{26} We find random matching to be an intuitive assumption in certain situations. For example, imagine that a number of equally prestigious law firms operate in different cities throughout the country. If the substantive dimensions of potential job offers are more-or-less equivalent at each firm, a graduate’s application and employment preference may be strongly influenced by their location preference. If these preferences are uncorrelated with the graduate’s transcript and ability, then matching between graduates and evaluators is essentially random. Similarly, it is not difficult to imagine that (for a certain subset of) high school graduates, college preference is strongly influenced by idiosyncratic tastes. Fraternity and sorority cultures, the characteristics of student bodies, location, and variety of courses or majors offered all influence the graduate’s college application decision. To the extent that these factors are independent of ability and transcript, matching is effectively random.

\textbf{5.2 OBSERVABLE GRADING POLICY}

Our analysis focuses on a situation in which an evaluator is able to observe a school’s grading policy directly. Equivalent results follow if a school’s grading policy is verifiable, but not observable.\textsuperscript{27} We believe that this assumption is reasonable because evaluators frequently have the capability to learn about grading policies at schools in a variety of ways. By collecting information about course content and “rigor,” as well as grade distributions, the evaluator can learn about a school’s grading policy. In some cases, schools directly supply information about grading policies to potential evaluators, including statistics on graduate’s transcripts.\textsuperscript{28}

Without this assumption, the grading game unravels. This is easiest to see in the context of the inflation-only model in Section 3. If the evaluator is unable to observe the probability that a low ability student is assigned grade $G$, she must infer it from the school’s equilibrium behavior. Suppose that the evaluator anticipates that a school assigns grade $G$ to a low-ability student with probability $\theta$, but can not observe the value of $\theta$ directly. If she sees grade $G$, the value of her posterior belief $g$ is determined by her prior belief, and the value of $\theta$ she anticipates the school will use. However, given the evaluator’s beliefs, a school could simply assign the grade of $G$ to all of its graduates, guaranteeing that the evaluator’s posterior belief about graduate ability is always $g$, and never zero. This is only consistent with Perfect Bayesian equilibrium if this is the behavior the evaluator initially expects. Thus, if the grading policy at the school is unobservable, grade inflation runs rampant: grades are maximally inflated and minimally informative. Nonetheless, as we show in Section 4.4, even this extreme situation may be better than the fully-revealing benchmark.

\textsuperscript{26}If students from each school are matched randomly to compete for a prize, then the school’s goal is to maximize the probability that a randomly selected graduate generates a posterior belief, higher than the posterior belief generated by the other graduate. Thus, the single graduate in the model presented in the paper, could be interpreted as a school’s randomly chosen graduate.

\textsuperscript{27}If a school’s grading policy is verifiable, but not observable, the school could potentially try to avoid disclosing its grading policy to the evaluator. If the evaluator believes that whenever a school does withhold its grading policy, the graduate’s transcript would reveal him to be low ability for certain, then a school would never withhold its grading policy from the evaluator (except perhaps in this instance).

\textsuperscript{28}As part of an assessment of its grading policies, the reviewers recommended that The University of North Carolina, Chapel Hill put the following clarifying quotation on its transcripts “The University of North Carolina at Chapel Hill strictly monitors its grading system in order to insure fairness and consistency both across units and over time. Therefore, the grades on this transcript reflect an overall grade average of 2.6-2.7. Special care should be taken in comparing grades on this transcript with grades from colleges and universities that have not controlled grade inflation. See the distribution of grades on the back of this transcript.”
5.3 QUADRATIC COSTS

In order to derive relatively simple closed-form solutions for all variables of interest, we have focused on the case in which the cost of improving expected student ability is quadratic. In the body of the paper, we have described the intuition, as much as possible, only in terms of marginal benefit; the reasoning and intuition in the paper therefore also apply for other strictly convex cost functions. While strict convexity is a standard assumption, we also point out that if costs were linear, the three stage game would be essentially outcome-equivalent to a setting with uninformative grades.

5.4 STUDENT EFFORT

In the analysis we present, the student is a passive participant in the learning process: only a school’s level of investment affects his likely educational outcome. It is natural to think that students actively participate in their own education by exerting effort. Student effort is consistent with the model we have presented thus far, as long as it is observable. In this case, part of a school’s investment in quality may include costly programs to directly monitor student effort and apply pressure to shirking students. However, if student effort is unobservable, monitoring and enforcing effort targets is difficult. In this case, a student’s effort choice is determined strategically, and is influenced by the school’s grading policy. Because the strategic grading equilibrium exhibits grade inflation, allowing strategic grading may undermine student effort.

To address this concern, we have solved a version of the model that features unobservable student effort. Here, we describe the model and the main results.\(^\text{29}\)

We augment our model by incorporating two strategic students, one for each school. In the first stage of the game, school investments and student effort are chosen simultaneously. These choices determine each student’s likely ability. Let \(q_i\) represent the probability that the graduate of school \(i\) is high ability (as in the models given so far), \(s_i\) represents school \(i\)'s investment, and \(t_i\) represent student \(i\)’s effort. We focus on the following specification:

\[
q_i = \omega s_i + (1 - \omega)t_i
\]

Thus, the probability that graduate \(i\) is high ability is a weighted sum of school investment and student effort, where the weight on school investment is given by \(\omega \in [0, 1]\). Each choice of school investment and student effort is associated with a quadratic cost function, with a potentially different marginal cost. As in the previous model, school investment is revealed publicly at the end of the first stage. However, each student’s effort is not observed by any other party. Both schools and the evaluator forecast each student’s effort during the grading and evaluation stages, and in equilibrium these forecasts are correct.

Unsurprisingly, when \(\omega = 0\) (so that educational outcomes are determined solely by unobserved student effort), noisy grading is associated with lower student effort and achievement compared to the fully-informative benchmark. In this case, allowing noisy grading is unambiguously bad for welfare, because it reduces both effort and grading policy informativeness. As we have already shown, however, when \(\omega = 1\), the equilibrium with strategic grading may exhibit greater investment by schools than in the fully-revealing benchmark, leaving the evaluator better off. The equilibrium is continuous in \(\omega\), so the qualitative implications hold for values of \(\omega\) near the extremes.\(^\text{30}\) Therefore, if educational outcomes are primarily determined by (unobservable) student effort, educational outcomes and evaluator welfare are reduced when noisy grading is permitted.\(^\text{31}\) If educational outcomes are primarily determined by (observable) school investment in education quality, the

\(^{29}\)Details are available upon request.

\(^{30}\)The solution cannot be written in closed form unless \(\omega = 0\) or \(\omega = 1\).

\(^{31}\)This effect may still be tempered by increased school investment.
main qualitative results of our analysis continue to hold, even when students contribute unobserved effort into the education process.

6 CONCLUSION

We use a sequential model of school competition, featuring endogenous investment, grading, and evaluation, to study the interaction between a school’s investment decision and its choice of grading policy. When schools control information about student ability, they choose grading strategies that do not perfectly reveal student types to evaluators. However, they also invest more to increase education quality than in a benchmark case where all information about ability is observed by evaluators. The increased investment has the potential to offset the cost of noisy grading.

Viewed from a broad perspective, our analysis considers competing institutions that make (real) investments to improve outcomes and also exert influence over the release of information about these outcomes. While our focus is on competition between schools, similar considerations are important in a variety of other settings. For example, a division within a firm may exert effort to develop a prototype, and simultaneously control the way in which the prototype is tested prior to undertaking a development decision. Competing for promotion, a number of employees may exert effort to improve their output, while simultaneously controlling the flow of information to superiors within the firm. In finance, firms make investments and exert effort to generate profits for shareholders, but have the capability to influence the information available to investors about their performance by controlling the way in which earnings are disclosed. Producers of consumer products invest to develop new product lines or models, but they may limit the availability of information about new product attributes prior to their release.

In the context of education, our results have novel implications for both the policy discussion and common industry practices. We highlight a potential benefit of grade inflation—that it leads to greater investment in education quality—and show that the benefit can dominate the costs associated with information loss. This suggests that grade inflation may not be as bad for welfare as popular discussion suggests. In fact, it may lead to better outcomes than would arise if student ability is perfectly observed. By implication, the results suggest that policies or practices that convey information about graduate ability can reduce investment in the quality of education, decreasing student ability. Industry licensing exams and university or graduate school entrance exams provide employers and universities with information about applicant ability. To the extent that performance on these tests is an indicator of ability, graduate school entrance exams (such as the LSAT, GRE, MCAT and GMAT) and state licensing exams in various industries may undermine the incentives of colleges and universities to invest in quality. College entrance exams like the SAT and ACT as well as state assessment exams may may have similar unintended consequences for primary and secondary education. Paradoxically, the use of such exams may reduce investment in education quality and graduate ability.

REFERENCES


Popov, Sergey, and Dan Bernhardt. forthcoming. “University Competition, Grading Standards and Grade Inflation.” _Economic Inquiry_.


A APPENDIX

A.1 BENCHMARK WITH FULLY-REVEALING GRADES

In the fully revealing benchmark, schools are required to use grading policies that completely reveal the ability of the graduate. As described in text, a fully-revealing grading policy generates a Bernoulli posterior: \( Pr(\Gamma_i = 1) = q_i \) and \( Pr(\Gamma_i = 0) = 1 - q_i \). Thus, at the grading stage, for fixed investment levels \((q_a, q_b)\) expected payoffs are

\[
u_i(q_a, q_b) = q_i(1 - \frac{q_j}{2}) + \frac{1}{2} \left( 1 - q_i \right) \left( 1 - q_j \right) = \frac{1}{2} (1 + q_i - q_j) \]

\[
u_e(q_a, q_b) = 1 - (1 - q_a)(1 - q_b).
\]

The marginal benefit of investment is constant at \( 1/2 \) at each school. Each school's investment level satisfies its first order condition: \( 1/2 = 2/\rho_i^2 \). Thus, each school has a dominant strategy to invest \( q_i = \rho_i^2 / 4 \).

The associated expected payoff for each school and evaluator is given by

\[
u_i = \frac{1}{2} (1 + \frac{\rho_j^2 - \rho_i^2}{4}) \quad \text{and} \quad \nu_e = 1 - \left( 1 - \frac{\rho_j^2}{4} \right) \left( 1 - \frac{\rho_i^2}{4} \right).
\]

A.2 GRADE INFLATION ONLY (SIMPLE MODEL)

The posterior belief that a graduate with a bad grade is high ability equals 0, since no high ability students receive bad grades. The posterior belief, \( g_i \), that the graduate \( i \) is high ability given a good grade ranges from \( g_i = q_i \) when \( \theta_i = 1 \) to \( g_i = 1 \) when \( \theta_i = 0 \). For all levels of grade inflation \( \theta_i \in [0, 1] \), Bayes’ rule provides a one-to-one mapping between \( \theta_i \) and \( g_i \), with \( g_i = q_i/(q_i + (1 - q_i)\theta_i) \), and \( \partial g_i / \partial \theta_i < 0 \).

The expected payoff of school \( i \) is therefore given by (1). The following lemmas describe the structure of the second stage equilibrium following for combination of investments \( q_a \geq q_b \) in the first stage.

**Lemma A.1** If \( q_b \geq 1 - q_a \) then the unique Nash equilibrium of the second stage game is \( q_a = q_b = 1 \).

**Proof.** If school \( j \) chooses \( g_j = 1 \), then the best possible deviation from \( g_i = 1 \) is \( g_i = q_i \). By choosing this deviation, school \( i \) assures that if the other school's graduate receives a \( B \) and is thus revealed to be low ability, school \( i \) payoff is 1. Thus, \( q_a = q_b = 1 \) is a Nash equilibrium if and only if for each school \( i \)

\[q_i(1 - \frac{q_j}{2}) + \frac{1}{2} \left( 1 - q_i \right) \left( 1 - q_j \right) \geq 1 - q_j \iff q_i + q_j \geq 1\]

Thus, the equilibrium of the second stage is fully revealing if and only if \( q_b \geq 1 - q_a \). 

**Lemma A.2** If \( \frac{1}{2}(1 - q_a) < q_b < 1 - q_a \) then the mixed strategy Nash equilibrium is as follows:

\[
g_a = \begin{cases} 
G \sim F(x) = \frac{q_a}{x^2 - q_a^2} & \text{with prob } \ 1 - \phi_1 - \phi_2 = 1 - \frac{2q_b}{q_a + q_b} \\
1 & \text{with prob } \phi_1 = \left( \frac{2q_b}{q_a + q_b} \right) \left( 1 - \frac{q_a + 2q_b - 1}{q_a(q_b + q_b)} \right)
\end{cases}
\]

\[
g_b = \begin{cases} 
G \sim F(x) = \frac{x^2 - q_a^2}{4(1 - q_a)(1 - q_a - q_b)} & \text{with prob } \lambda = 1 - \frac{2q_b}{q_a + q_b} \\
1 & \text{with prob } 1 - \lambda = \frac{2q_b}{q_a + q_b}
\end{cases}
\]
Proof. Note, all probabilities are positive and sum to one, and the density of $G$ is given by $f(x) = x/(2(1-q_b)(1-q_a-q_b))$. The support of $G$ is $[q_a, 2-q_a-2q_b]$, and for the parameters of the proposition, the top of the support is in $[0,1]$. To show that the proposed strategies constitute a mixed strategy Nash equilibrium, we verify that each school is indifferent among all pure strategies inside the support and that no pure strategy outside the support delivers a better expected payoff against the mixed strategy of the other player.\footnote{We follow a similar approach in the proof to the next lemma as well. Alternative derivation of the equilibrium from the indifference conditions, which also shows uniqueness, is available for both equilibrium cases upon request.}

School $B$'s expected payoff from pure strategy $p$ in the support of its mixed strategy:

$$u_b = \begin{cases} \frac{q_b}{p} (1 - \phi f_p^{q_a-2q_b} f(s) \frac{q_a}{s} ds - \phi f_{2q_a}(s) \frac{q_a}{s} ds) + \frac{1}{2} (1 - E[\frac{q_a}{g_a}]) (1 - \frac{q_b}{p}) & \text{if } p \in [q_a, 2-q_a-2q_b] \\ q_b (1 - \frac{2q_g}{2}) + \frac{1}{2} (1 - E[\frac{q_a}{g_a}]) (1 - q_b) & \text{if } p = 1 \end{cases}$$

Substitution and simplification gives:

$$u_b = \begin{cases} \frac{q_b}{p} \frac{q_a^2 + q_a q_b}{(q_a + q_b)^2} + \frac{1}{2} \frac{2q_b^2}{(q_a + q_b)^2} (1 - \frac{q_b}{p}) & \text{if } p \in [q_a, 2-q_a-2q_b] \\ q_b (1 - \frac{q_a}{2} \frac{2q_b}{q_b + q_a} (1 + \frac{2q_b}{q_b + q_a})(1 - q_b) + \frac{1}{2} \frac{2q_b^2}{(q_a + q_b)^2} (1 - q_b) & \text{if } p = 1 \end{cases}$$

Further simplification gives $u_b = q_b/(q_a + q_b)$ in both cases. Thus all pure strategies in the support of $B$'s mixed strategy give the same expected payoff. Choosing any pure strategy $\hat{g} \in (2-q_a-2q_b, 1)$ is dominated by choosing $g = 2-q_a-2q_b$, because the probability of winning is the same for both pure strategies, but $g$ is more likely to generate a good realization. It is also straightforward to verify that choosing $q_a$ is dominated by the equilibrium mixed strategy. Thus all pure strategies in the support of $B$'s mixed strategy give the same expected payoff against $A$'s mixed strategy, and no strategy outside the support gives $B$ a higher payoff. Thus, $B$'s mixed strategy is a best response to $A$'s mixed strategy.

A symmetric analysis applies to school $A$. Simplifying the utility function gives $u_a = q_a/(q_a + q_b)$, and by a symmetric argument as above, one can show that $A$ mixed strategy is a best response to $B$'s mixed strategy. \quad \blacksquare

Lemma A.3 If $q_b \leq \frac{1}{2} (1 - q_a)$ then the mixed strategy Nash equilibrium is as follows:

$$g_a = \begin{cases} q_a & \text{with probability } 1 - \phi = 1 - \frac{2q_b}{q_b + q_a} \\ G \sim F(x) = \frac{x^2 - q_a^2}{4q_a(q_a + q_b)} & \text{with probability } \phi = \frac{2q_b}{q_b + q_a} \end{cases}$$

$$g_b = \begin{cases} G \sim F(x) = \frac{x^2 - q_a^2}{4q_a(q_a + q_b)} \\ q_b & \text{with probability } 1 - \phi = 1 - \frac{2q_b}{q_b + q_a} \end{cases}$$

Proof. Note, all probabilities are positive and sum to one, and the density of $G$ is given by $f(x) = x/(2q_b(q_a + q_b))$. The support of $G$ is $[q_a, q_a + 2q_b]$, and for the parameters of the proposition, the top of the support is in $[0,1]$. As before, we establish indifference between all pure strategies played with positive probability, and show that no pure strategy outside of the mixing distribution results in higher expected payoffs.

Consider school $B$'s expected payoff from a pure strategy $p \in [q_a, q_a + 2q_b]$ in the support of its mixed strategy:

$$u_b = \frac{q_b}{p} (1 - \phi f_p^{q_a+2q_b} f(s) \frac{q_a}{s} ds + \frac{1}{2} (1 - E[\frac{q_a}{g_a}]) (1 - \frac{q_b}{p})$$
Substitution and simplification gives:

\[ u_b = \frac{q_b}{p} \left(1 - \frac{2q_b}{q_a + q_b} q_a \frac{q_a + 2q_b - p}{2q_b} - \frac{1}{2} \left(1 - \frac{2q_b}{q_a + q_b} \frac{q_a}{q_a + q_b} - \left(1 - \frac{2q_a}{q_a + q_b}\right)\right) \right) \left(1 - \frac{q_b}{p}\right) \]

Further simplification gives \( u_b = q_b / (q_a + q_b) \). Following the same argument as in the proof to the previous lemma, one can show that \( B \) mixed strategy is a best response to \( A \)'s mixed strategy.

Consider school \( A \)'s expected payoff from a pure strategy \( p \) in the support of its mixed strategy:

\[ u_a = \frac{q_a}{p} \left(1 - f_{q_a + 2q_b} f(s) \frac{q_a}{q_a + q_b} ds + \frac{1}{2} \left(1 - E\left[ q_a \right]\right) \right) \left(1 - \frac{q_a}{p}\right) \] if \( p \in [q_a, q_a + 2q_b] \)

Substitution and simplification gives \( u_a = q_a / (q_a + q_b) \). Following the same argument as in the proof to the previous lemma, one can show that \( A \) mixed strategy is a best response to \( B \)'s mixed strategy.

**Evaluator payoff with fixed investment.** Here we calculate the evaluator payoff for symmetric investment levels for symmetric fixed investments \( q_a = q_b = q \). Because we consider symmetric schools, the equilibrium of the investment stage will also be symmetric. This calculation will therefore facilitate the comparison of evaluator payoff under inflation to the evaluator payoff in the fully-revealing benchmark. Given the realizations of \((g_a, g_b)\) generated by the schools' equilibrium mixed strategies, define \( g_m = \max\{g_a, g_b\} \) and \( g_n = \min\{g_a, g_b\} \). The evaluator's expected payoff for a particular combination of \((g_m, g_n)\) is given by

\[ q \frac{g}{g_m} + \left(1 - \frac{q}{g_m}\right) g_n \frac{q}{g_n} = q(2 - q) \]

If the graduate of the school with less inflated grading policy \( g_m \) generates a good transcript, an event that happens with probability \( \frac{q}{g_m} \), then the evaluator will accept that graduate, giving the evaluator payoff \( g_m \). If the graduate generates a bad transcript, then the evaluator knows for sure that he is low ability. If the other graduate generates a good transcript realization, \( \frac{q}{g_n} \) the evaluator accepts the graduate of the school with the more-inflated grading policy, giving the school expected payoff \( g_n \).

In order to evaluate the evaluator payoff, we need to determine the expected value of the inverse of the maximum order statistic from the equilibrium mixed strategies.

Case I: \( q \leq \frac{1}{4} \). When \( q_a = q_b = q \leq \frac{1}{4} \), the equilibrium mixed strategy of each school is to randomize over support \([q, 3q]\) using distribution function \( F(x) = \frac{x^2 - q^2}{8q^2} \) with corresponding density \( f(x) = \frac{x}{4q^2} \). The expectation in question, \( E\left[ \frac{1}{g_m} \right] \), is therefore

\[ \int_q^{3q} 2 \left(\frac{1}{x}\right) \left(\frac{x^2 - q^2}{8q^2}\right) \frac{x}{4q^2} dx = \frac{5}{12q} \]

Thus the evaluator expected payoff is \( q(2 - q) \frac{5}{12q} = 19/12q \).

Case II: \( \frac{1}{3} \leq q \leq \frac{1}{2} \). Here, the equilibrium mixed strategy of each school is as follows. With probability \( \phi = \frac{3q^2 - 1}{2q^2} \) choose \( g = 1 \). With probability \( 1 - \phi \) randomize over support \([q, 2 - 3q]\) using distribution function \( F(x) = (x^2 - q^2) / (4(1 - q)(1 - 2q)) \) with corresponding density \( f(x) = x / (2(1 - q)(1 - 2q)) \). Thus,

\[ \int_q^{3q} \left(\frac{x^2 - q^2}{8q^2}\right) \frac{x}{4q^2} dx = \frac{5}{12q} \]

The density of the maximum order statistic \( g_m \) is \( 2f(x)F(x) \).
point is feasible. The first order condition is given by:
This function is differentiable and concave. The unique critical point of function \( p \) is \( q = \frac{1}{2} \). However, because \( p < \sqrt{2} \) implies that \((p - \sqrt{2})(p + 2\sqrt{2}) < 0\), which may rearranged to give \( \frac{1}{4}p^2 < 1 - \frac{\sqrt{2}}{4}p \), implying that the critical point is less than the smallest feasible value. Thus, the maximum occurs at the left endpoint, which gives a payoff of

\[ \frac{1}{2}(1 + q_{i} - \frac{\sqrt{2}}{4}p) - \frac{q_{j}^2}{p^2} \]

This function is differentiable and concave. The best deviation greater that \( 1 - \frac{\sqrt{2}}{4}p \) is given by the solution to the following maximization problem:

\[ \max_{q_{i}} \frac{1}{2}(1 + q_{i} - \frac{\sqrt{2}}{4}p) - \frac{q_{j}^2}{p^2} \quad \text{subject to} \quad q_{i} > 1 - \frac{\sqrt{2}}{4}p. \]

This payoff function is continuous, and is differentiable, everywhere except for possibly \( q_{i} = q_{j} \). The Nash equilibrium of the investment stage is given in the following lemma.

**Lemma A.4** The Nash equilibrium of the first stage game is \( q_{i} = q_{j} = \frac{\sqrt{2}}{4}p \).

**Proof.** We maintain the assumption that \( \rho \leq \sqrt{2} \). Suppose \( q_{j} = \frac{\sqrt{2}}{4}p \). The best deviation greater that \( 1 - \frac{\sqrt{2}}{4}p \) is given by the solution to the following maximization problem:

\[ \max_{q_{i}} \frac{1}{2}(1 + q_{i} - \frac{\sqrt{2}}{4}p) - \frac{q_{j}^2}{p^2} \quad \text{subject to} \quad q_{i} > 1 - \frac{\sqrt{2}}{4}p. \]

This function is differentiable and concave. The unique critical point of function \( \frac{1}{2}(1 + q_{i} - \frac{\sqrt{2}}{4}p) - \frac{q_{j}^2}{p^2} \) is equal to \( \frac{1}{4}p^2 \). However, because \( p \leq \sqrt{2} \), this point is infeasible. To see this, note that \( p < \sqrt{2} \) implies that \((p - \sqrt{2})(p + 2\sqrt{2}) < 0\), which may rearranged to give \( \frac{1}{4}p^2 < 1 - \frac{\sqrt{2}}{4}p \), implying that the critical point is less than the smallest feasible value. Thus, the maximum occurs at the left endpoint, which gives a payoff of

\[ \frac{1}{2}(1 + (1 - \frac{\sqrt{2}}{4}p) - \frac{\sqrt{2}}{4}) - \frac{(1 - \frac{\sqrt{2}}{4}p)^2}{p^2} = \frac{\sqrt{2}}{4}p(2\sqrt{2} - \rho)(\rho^2 + \frac{\sqrt{2}}{4}p - 1) \]

If no deviation in the region \( q_{i} \leq 1 - \frac{\sqrt{2}}{4}p \) dominates \( q_{i} = \frac{\sqrt{2}}{4}p \), and the payoff associated with playing \( q_{i} = \frac{\sqrt{2}}{4}p \) as a response to \( q_{j} = \frac{\sqrt{2}}{4}p \) gives a higher payoff than the above expression, then \( q_{i} = q_{j} = \frac{\sqrt{2}}{4}p \) is the unique Nash equilibrium. Consider now the best deviation in the region \( q_{i} \leq 1 - \frac{\sqrt{2}}{4}p \), defined by the following maximization problem:

\[ \max_{q_{i}} \frac{q_{i}}{q_{i} + \frac{\sqrt{2}}{4}p} - \frac{q_{j}^2}{p^2} \quad \text{subject to} \quad q_{i} \leq 1 - \frac{\sqrt{2}}{4}p \]

This function is differentiable and concave. The max therefore occurs at the critical point, provided this point is feasible. The first order condition is given by:

\[ \frac{1}{p^2(q_{i} + \frac{\sqrt{2}}{4}p)^2} \frac{\sqrt{2}}{4}p^3 - \frac{1}{4}p^2q_{i} - \sqrt{2}pq_{i}^2 - 2q_{i}^3. \]
Substituting in $q_i = \frac{\sqrt{2}}{4} \rho$ reveals that the derivative is zero, and therefore that, if feasible, this level of $q_i$ is the best value less than $1 - \frac{\sqrt{2}}{4} \rho$. In fact, because $\rho \leq \sqrt{2}$, it follows that $\frac{\sqrt{2}}{4} \rho \leq \frac{1}{2}$ and thus $\frac{\sqrt{2}}{4} \rho \leq 1 - \frac{\sqrt{2}}{4} \rho$, so this value is feasible. Finally, we compare the payoff of choosing $q_i = \frac{\sqrt{2}}{4} \rho$ (which simplifies to $\frac{1}{2} - \frac{(\sqrt{2})^2 \rho^2}{4 \rho^2} = \frac{3}{8}$) to the best deviation in the other interval. The difference in the payoffs is therefore

$$3 \frac{8}{4} - \frac{\sqrt{2}}{4} \rho (2 \sqrt{2} - \rho) (\rho^2 + \frac{\sqrt{2}}{4} \rho - 1) = \frac{\sqrt{2}}{4} \rho (\rho + \sqrt{2}) (\rho - \sqrt{2})^2 \geq 0$$

Thus, whenever $\rho \leq \sqrt{2}$, choosing $q_i = q_j = \frac{\sqrt{2}}{4} \rho$ is a Nash equilibrium. ■

Note that if $\sqrt{2} < \rho < 2$, then the Nash equilibrium involves both schools choosing fully-revealing grading policies. We leave formal consideration of these alternative parameter values to the sections involving unrestricted grading policies.

A.2.1 Inflation only (simple model) versus fully-revealing grading

**Investment comparison.** In the equilibrium with symmetric abilities ($\rho_\alpha = \rho_\beta = \rho \leq \sqrt{2}$) above, schools invest $\frac{\sqrt{2}}{4} \rho$. For these parameters, school investment in the fully-revealing benchmark is $\frac{\sqrt{2}}{4}$. Whenever $\rho \leq \sqrt{2}$, investment is greater in the inflationary equilibrium.

**Evaluator payoff comparison with fixed investment.** Here we show that, as in the general model, when investments are fixed, the evaluator receives a higher payoff when schools are constrained to use fullyrevealing grading policies.

The difference between the evaluator expected payoff in the fully-revealing benchmark and the inflation equilibrium is given by

$$1 - (1-q)^2 - \frac{19}{12} q = \frac{1}{12} q (12q - 5)$$

$$1 - (1-q)^2 - \frac{19}{12} q (1 - 9q + 27q^2 - 8q^3) = \frac{1}{4} q (1 - 2q)^2 (q - \frac{\sqrt{2}}{6} - \frac{5}{6}) q + \frac{\sqrt{2}}{6} - \frac{5}{6}$$

if $q > \frac{1}{2}$

Casual inspection reveals that the payoff from the revealing benchmark $1 - (1-q)^2$ is greater than the equilibrium payoff, whenever the equilibrium is not itself fully-revealing, $q \leq \frac{1}{2}$.

**Evaluator payoff comparison with endogenous investment.** Here we show that, as in the general model, when investments are endogenous, the evaluator receives a (weakly) higher payoff when schools are permitted to inflate grades. We consider $\rho_\alpha = \rho_\beta = \rho \leq \sqrt{2}$. In this case equilibrium investment with inflation is given by $\frac{\sqrt{2}}{4} \rho$, while fully revealing investment is $\frac{\sqrt{2}}{4}$.

The difference between the evaluator expected payoff in the fully-revealing benchmark and the inflation equilibrium is given by

$$1 - (1-q)^2 - \frac{19}{12} \frac{\sqrt{2}}{4} \rho = \frac{1}{12} q (12q - 5)$$

$$1 - (1-q)^2 - \frac{19}{12} \frac{\sqrt{2}}{4} (1 - 9 \frac{\sqrt{2}}{4} \rho + 27 (\frac{\sqrt{2}}{4} \rho)^2 - 8 (\frac{\sqrt{2}}{4} \rho)^3)$$

if $\rho \leq \frac{3}{5} \sqrt{2}$

Plotting these payoff functions clearly shows that when investment is endogenous, evaluator payoff is higher when grade inflation is allowed.
A.3 GAME WITH GENERAL GRADING POLICIES

In the text we represent a grading policy by the distribution of the posterior belief that will be induced by the grading policy. Here we explain this construction in detail. Recall that for \((H, L)\), to be an admissible grading policy, each random variable \(H, L\) can have a countable number of mass points, and otherwise admits a differentiable density function \(f_H(x), f_L(x)\). Any random variable that satisfies these conditions is valid. In addition, the likelihood ratio of a grading policy must be monotone. Let \(T \in \{H, L\}\). Also \(m_T^j\) represent a mass point of random variable \(T\) and \(\mu^j_T = \Pr(H = m^j_T)\). The density function of random variable \(T\) is given by

\[
p_T(x) = f_T(x) + \delta(x - m^j_T)\mu^j_T
\]

where \(\delta(x)\) represents the Dirac \(\delta\)-function. Of course, because the density contains the Dirac delta function, it is not a proper function. This statement should be interpreted to mean that the cumulative distribution function of \(T\) is \(P_T(x) = F_T(x) + \sum_j S(x - m^j_T)\mu^j_T\), where \(F_T(x)\) is the integral of \(f_T(x)\), and \(S(x)\) is the step (or Heaviside) function. A graduate's transcript is also a random variable, \(X\), whose distribution is determined by the school grading policy. This also implies that the posterior belief is a random variable:

\[
F_X(x) = \Pr(X \leq x) = qP_H(x) + (1 - q)P_L(x)
\]

\[
f_X(x) = qp_H(x) + (1 - q)p_L(x)
\]

Transcripts convey information about student quality. Because the true ability of the graduate is unknown, each possible realization of his transcript induces a posterior belief that the graduate is high ability. This posterior belief depends on the prior belief \(q\), (determined by school investment), the realization of the graduate’s transcript \(x\), and the school’s grading policy \((H, L)\). The evaluator’s posterior belief that the graduate is high ability (derived from Bayes’ rule) is given by

\[
\Pr(t = h|x) = \frac{qp_H(x)}{qp_H(x) + (1 - q)p_L(x)}
\]

At the time the school designs its grading policy, the realization of the transcript is uncertain. The transcript \(X\) is a random variable, whose distribution is determined by the school grading policy. This also implies that the posterior belief is a random variable:

\[
\Gamma = \Pr(t = h|X)
\]

Once a school designs its grading policy, but before the student’s transcript is realized, the value of the posterior belief is a random variable \(\Gamma\). This random variable \(\Gamma\) is thus the \textit{ex ante} value of the posterior belief. The random variable \(\Gamma\) is valid, has support confined to the unit interval, and (from the Law of Total Expectation) has expected value equal to the prior: \(E[\Gamma] = q\). In the text we claim that, these are the only substantive restrictions on random variable \(\Gamma\). Here we provide the proof:

**Proof to Lemma 4.1.** Let \(\Gamma\) be a random variable that satisfies the assumptions of the Lemma. \(\Gamma\)

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\[\text{To see that } \Gamma \text{ is valid, consider the following. A mass point of } \Gamma \text{ must be in } M_G \cap M_B \cup \{0, 1\}, \text{ a finite set. Any realization outside of } I_G \cap I_B \text{ leads to posterior belief } 0 \text{ or } 1. \text{ All non-mass points inside of } \Gamma \text{ are generated by signal realizations inside } I_G \cap I_B. \text{ Because } f_G(x) \text{ and } f_B(x) \text{ are continuous on this interval, the posterior belief is also continuous on this interval. By the intermediate value theorem, if two posteriors are generated by a realization inside the support, then all posteriors in between are also generated by a signal realization inside } I_G \cap I_B. \text{ Continuity of the density over the interval of support follows from differentiability and monotonicity of the likelihood ratio.} \]
therefore has some density function \( g(x) \) of the type given in equation (2). Consider two random variables \( H \) and \( L \) with densities \( h(x), l(x) \) given by the following:

\[
h(x) = \frac{xg(x)}{q} \quad \text{and} \quad l(x) = \frac{(1-x)g(x)}{1-q}
\]

Observe that the supports of \( H \) and \( L \) are identical to the support of \( \Gamma \). Also observe that because the expected value of \( \Gamma \) is equal to \( q \), both \( h(x) \) and \( l(x) \) are proper density functions. Consider next the posterior belief associated with realization \( x \)

\[
Pr(t = h|x) = \frac{qh(x)}{qh(x) + (1-q)l(x)} = \frac{xg(x)}{xg(x) + (1-x)g(x)} = x
\]

Thus the posterior belief associated with any transcript realization is equal to the realization of the transcript itself. The likelihood ratio is monotone, as the posterior belief is monotonic in the realization. The density of the transcript realization is given by

\[
qh(x) + (1-q)l(x) = g(x)
\]

For the constructed grading policy, the posterior belief is equal to the realization of the transcript, and the transcript has density \( g(x) \). Thus for the grading policy constructed, \( \Gamma \) is the ex ante posterior belief.

### A.3.1 Stage two equilibrium

**Proof to Lemma 4.2.** We verify that the proposed strategies are an equilibrium. Uniqueness is proved separately in the online appendix. Suppose school \( A \) invests \( q_a \), while school \( B \) invests \( q_b \) and \( q_a \geq q_b \).

CASE I: \( q_a \leq 1/2 \). Consider the best response of school \( B \) to the strategy of school \( A \). Any admissible strategy on the part of school \( B \) is a random variable \( G_b \) that can be represented in the following way:

\[
G_b = \begin{cases} 
G_1 & \text{with prob } p \\
G_2 & \text{with prob } 1-p.
\end{cases}
\]

Here \( G_1 \) is any valid random variable over support \([0, 2q_a]\) density \( g_1(x) \) and expectation \( \bar{g}_1 \), and \( G_2 \) is any valid random variable over support \([2q_a, 1]\) density \( g_2(x) \) and expectation \( \bar{g}_2 \). If a mass point exists at \( 2q_a \), then we include the mass point in random variable \( G_1 \). The constraint on the mean of \( G_b \) implies that \( \bar{g}_1 p + (1-p)\bar{g}_2 = q_b \), or equivalently

\[
\bar{g}_1 = \frac{q_b - (1-p)\bar{g}_2}{p}.
\]

The expected payoff of school \( B \) from any random variable \( G \) against \( \Gamma_a \) is given by

\[
p \int_0^{2q_a} g_1(x) \frac{x}{2q_a} dx + (1-p)
\]

which simplifies to

\[
\frac{q_b}{2q_a} - (1-p)(\frac{\bar{g}_2}{2q_a} - 1).
\]
If any probability mass exists in the interval \((2qa, 1]\) then \(\bar{g}_2 > 2qa\), and thus in any best response \(p = 1\). Furthermore, any random variable \(G_b\) for which \(p = 1\) gives the same expected payoff \(u_b = \frac{q_b}{2q_a}\), and is therefore a best response. Because the support of \(\Gamma_b\) is \([0, 2qa]\), it is a best response.

Next, we show that \(\Gamma_a\) is a best response to \(\Gamma_b\). Any admissible best response on the part of school \(A\) is a random variable \(G_a\) that can be represented in the following way:

\[
G_a = \begin{cases} 
G_1 \text{ with prob } p \\
G_2 \text{ with prob } 1 - p
\end{cases}
\]

The constraint on the mean of \(G_a\) implies that \(\bar{g}_1p + (1 - p)\bar{g}_2 = qa\), or equivalently

\[
\bar{g}_1 = \frac{qa - (1 - p)\bar{g}_2}{p}.
\]

The expected payoff of school \(B\) from any random variable \(G\) against \(\Gamma_a\) is given by

\[
p(1 - \frac{q_b}{qa} + \frac{q_b}{qa} \int_0^{2qa} g_1(x) \frac{x}{2qa} dx) + (1 - p)
\]

which simplifies to

\[
1 - \frac{q_b}{2qa} - (1 - p) \frac{q_b}{2qa} (\bar{g}_2 - 2qa).
\]

Following the same reasoning as with \(\Gamma_b\) above, one establishes that \(\Gamma_a\) is also a best response.

CASE II: \(qa > \frac{1}{2}\). Any admissible strategy on the part of school \(i\) is a random variable \(G_i\) that can be represented in the following way:

\[
G_i = \begin{cases} 
G_1 \text{ with prob } p_1 \\
G_2 \text{ with prob } p_2 \\
1 \text{ with prob } p_3
\end{cases}
\]

where \(p_1 + p_2 + p_3 = 1\). Here \(G_1\) is any valid random variable over support \([0, 2(1 - qa)]\) density \(g_1(x)\) and expectation \(\bar{g}_1\), and \(G_2\) is any valid random variable over support \([2(1 - qa), 1]\) density \(g_2(x)\) and expectation \(\bar{g}_2\). If a mass point exists at \(2(1 - qa)\) then we include the mass point in random variable \(G_1\). The constraint on the mean of \(G_i\) implies that

\[
\bar{g}_1 p_1 + p_2 \bar{g}_2 + p_3 = qi \rightarrow \bar{g}_1 = \frac{q_i - p_2 \bar{g}_2 - p_3}{p_1}
\]

The expected payoff of school \(B\) from any random variable \(G_b\) against \(\Gamma_a\) is given by

\[
p_1 \left(1 - \frac{1}{qa}\right) \int_0^{2(1 - qa)} g_1(x) \frac{x}{2(1 - qa)} dx + p_2 \left(\frac{1}{qa} - 1\right) + p_3 \left(\frac{1}{qa} - 1\right) + (2 - \frac{1}{qa}) \frac{1}{2}
\]

which simplifies to

\[
\frac{q_b}{2qa} - \frac{1}{2qa} (\bar{g}_2 - 2(1 - qa)) p_2.
\]

If any probability mass exists in the interval \((2(1 - qa), 1]\) then \(\bar{g}_2 > 2(1 - qa)\). The coefficient on \(p_2\) is
therefore negative. Thus in any best response, \( p_2 = 0 \). Furthermore, any random variable \( G_b \) for which \( p_2 = 0 \) gives the same expected payoff \( u_b = \frac{q_b}{q_a} \), and is therefore a best response. Because the support of \( \Gamma_b \) is \([0, 2(1 - q_a)] \cup 1\), it is a best response.

A symmetric approach establishes that the expected payoff to school \( A \) from any random variable \( G_a \) against \( \Gamma_b \) simplifies to

\[
(1 - \frac{q_b}{q_a}) + \frac{q_b}{q_a} \frac{1}{2} - \frac{1}{2q_a} (\hat{g}_2 - 2(1 - q_a)) p_2.
\]

A similar argument may be applied in the case of \( B \) as was applied to the case of \( B \) in order to establish that \( \Gamma_a \) is a best response. ■

**Equilibrium mimicry.** We briefly show that in equilibrium, school \( B \) underlying grading policy is identical to that of school \( A \), except for the inclusion of a transcript or set of transcripts that are assigned to low ability students only.

Suppose that school \( A \) utilizes an equilibrium grading policy \((H_a, L_a)\), where the density of \( H_a \) is \( h_a(x) \) and the density of \( L_a \) is \( l_a(x) \). Thus, posterior belief about ability at school \( A \) given transcript \( x \) is \( \frac{q_la(h_a(x))}{q_a h_a(x) + (1 - q_a)l_a(x)} \) and the density of the posterior is \( q_a h_a(x) + (1 - q_a)l_a(x) \).

In equilibrium, the distribution of posterior beliefs at both schools is identical, except that school \( B \) reveals a probability mass of \( 1 - \frac{q_b}{q_a} \) students to be low ability. In order to do so, school \( B \) assigns an additional grade (or set of grades) that we call \( F \). This grade is assigned only to low ability students; thus observing \( F \) reveals that the student is low ability. In equilibrium the probability that school \( B \) assigns \( F \) is \( 1 - \frac{q_b}{q_a} \). Therefore, the conditional probability that a student gets an \( F \) given that he is low ability is

\[
\phi = \frac{1 - \frac{q_b}{q_a}}{1 - q_b} = \frac{q_a - q_b}{q_a(1 - q_b)}.
\]

Suppose that school \( B \) grading policy is identical to school \( A \), except for the inclusion of the \( F \) grade, assigned to low ability students with conditional probability \( \phi \). The posterior belief about \( B \) graduate if the transcript is not \( F \) is given by

\[
\frac{q_b h_a(x)}{q_b h_a(x) + (1 - q_b)(1 - \phi)l_a(x)} = \frac{q_a h_a(x)}{q_a h_a(x) + (1 - q_a)l_a(x)}.
\]

Thus, school \( B \) equilibrium grading policy assigns an \( F \) to low ability students with conditional probability \( \phi \), and otherwise is identical to the grading policy of school \( A \).

**Equilibrium payoffs.** We calculate the equilibrium payoffs for the evaluator. Equilibrium payoffs for the schools have been calculated in the body of the proof of equilibrium.

**Corollary A.5 evaluator payoff**

- If \( q_a \leq \frac{1}{2} \) then evaluator expected payoff is \( u_e = q_a + \frac{1}{3} q_b \).
- If \( q_a > \frac{1}{2} \) then evaluator expected payoff is \( u_e = \frac{3q_a^4 - 7q_a^3 q_b + 12q_a^2 q_b - 6q_a q_b + q_b}{3q_a^3} \).

**Proof.** In either case, the evaluator’s payoff is given by the following expression:

\[
\frac{q_b}{q_a} E[\Gamma_{a}^{(2)}] + (1 - \frac{q_b}{q_a}) q_a
\]

Here, \( \Gamma_{a}^{(2)} \) represents the maximum order statistic from two draws of random variable \( \Gamma_a \). If school \( b \) does not reveal the student to be low ability for sure, then it uses the school \( A \) grading policy. In this case the evaluator payoff is the maximum draw. If, however, school \( b \) reveals its graduate to be low ability, the evaluator only receives the expected quality of a graduate from school \( A \).
CASE I: \( q_a \leq \frac{1}{2} \). In this case, \( \Gamma_a = U[0, 2q_a] \) and therefore, \( E[\Gamma_a^{(2)}] = \frac{4}{3} q_a \). Evaluating (3) gives expected evaluator payoffs

\[
\frac{q_b}{q_a} \left( \frac{4}{3} q_a \right) + (1 - \frac{q_b}{q_a}) q_a = q_a + \frac{1}{3} q_b
\]

CASE II: \( q_a > \frac{1}{2} \). Here, \( \Gamma_a = U[0, 2(1-q_a)] \) with probability \( 1/q_a - 1 \) and \( \Gamma_a = 1 \) otherwise. Thus, evaluating \( E[\Gamma_a^{(2)}] \) gives

\[
E[\Gamma_a^{(2)}] = 1 - \left( \frac{1}{q_a} - 1 \right)^2 + (1 - \frac{1}{q_a} - 1)^2 \left( \frac{4}{3} \right) (1 - q_a).
\]

Substituting and simplifying gives

\[
E[\Gamma_a^{(2)}] = \frac{1 - 6q_a + 12q_a^2 - 4q_a^3}{3q_a^2}.
\]

A.3.2 Stage one equilibrium

Here we derive the stage one equilibrium effort levels for both schools. The payoff function for each school \( i \in \{ \alpha, \beta \} \) is given by

\[
u_i(q_i, q_j) = \begin{cases} \frac{q_i}{2q_i} - \frac{q_i^2}{r_i} & \text{if } q_i \leq q_j \\ 1 - \frac{q_i}{2q_i} - \frac{q_i^2}{r_i} & \text{if } q_i > q_j \end{cases}
\]

This function is differentiable, continuous, (weakly) concave in \( q_i \). Therefore the unique maximum occurs wherever the derivative of the payoff function is equal to zero:

\[
\frac{du_i(q_i, q_j)}{dq_i} = \begin{cases} \frac{1}{2q_i} - \frac{q_i}{r_i} & \text{if } q_i \leq q_j \\ \frac{q_i}{2q_i} - \frac{2q_i}{r_i} & \text{if } q_i > q_j \end{cases}
\]

Suppose that in equilibrium \( q_a \geq q_b \). Such an equilibrium is described by the following first order conditions:

\[
\frac{1}{2q_a} - \frac{2q_b}{r_b} = 0 \quad \text{and} \quad \frac{q_b}{2q_a^2} - \frac{2q_a}{r_a^2} = 0
\]

Therefore we find that

\[
q_a = \frac{\sqrt{r_a r_b}}{2} \quad \text{and} \quad q_b = \frac{r_b^2}{2\sqrt{r_a r_b}}
\]

The difference \( q_\alpha - q_\beta = \frac{\sqrt{r_a r_b}}{2\sqrt{r_a r_b}} (\rho_a - \rho_b) \). Hence, for this combination to be an equilibrium it must be that \( \rho_a \geq \rho_b \), that is, school with the higher resource, \( \alpha \), invests more in equilibrium (thereby playing the role of school \( A \)). Thus

\[
q_\alpha = \frac{\sqrt{r_\alpha r_\beta}}{2} \quad \text{and} \quad q_\beta = \frac{r_\beta^2}{2\sqrt{r_\alpha r_\beta}}
\]

\[
u_\alpha = \frac{r_\beta}{4\rho_a} \quad \text{and} \quad u_\beta = 1 - \frac{3\rho_\beta}{4\rho_\alpha}
\]

We find the evaluator payoff by substituting the school effort levels into the evaluator payoff function. The two cases arise because \( \rho_a \rho_\beta \leq 1 \) implies \( q_a \leq \frac{1}{2} \) which influences the nature of the equilibrium that arises
in the second stage.

$$u_e = \begin{cases} \frac{(3\rho_a + \rho_\beta)\sqrt{\rho_a \rho_\beta}}{6\rho_a} & \text{if } \rho_a \rho_\beta \leq 1 \\
\frac{(3\rho_a^2 - 7\rho_a \rho_\beta - 24)\sqrt{\rho_a \rho_\beta} + 24\rho_a \rho_\beta + 8}{6\rho_a} & \text{if } \rho_a \rho_\beta > 1 \end{cases}$$

A.3.3 Fully-revealing versus strategic grading

Investment Comparison. Here we compare the investment of schools in the strategic grading benchmark to investment under fully-informative grading. The following inequalities define the conditions under which investment under strategic grading is higher than under fully revealing grading.

\[
\text{School } \alpha: \ \frac{\sqrt{\rho_a \rho_\beta}}{2} \geq \frac{\rho^2}{4} \iff \rho_\beta \geq \frac{\rho^3}{4} \\
\text{School } \beta: \ \frac{\rho^2}{2\sqrt{\rho_a \rho_\beta}} \geq \frac{\rho^2}{4} \iff 2 \geq \sqrt{\rho_a \rho_\beta}
\]

The last inequality is always satisfied given the assumption \( \rho_i \leq 2 \).

Payoff comparison with fixed investment. Here we show that for fixed investments the evaluator and school \( \beta \) expect higher payoff when grades must be fully-revealing, while school \( \alpha \) prefers the equilibrium without grading restrictions. We begin with the schools.

\[
\frac{1}{2}(1 + q_b - q_a) = \frac{q_b}{2q_a} + \frac{1}{2}(1 - \frac{q_b}{q_a})(1 - q_a)
\]

\[
\frac{1}{2}(1 + q_a - q_b) = 1 - \frac{q_b}{2q_a} - \frac{1}{2}(1 - \frac{q_b}{q_a})(1 - q_a)
\]

On the left hand sides we have the school payoffs under full-information, while on the right we have the payoffs in equilibrium without restriction. The school with less investment does better under fully informative transcripts while the school that invests more does worse. For the evaluator,

\[
1 - (1 - q_a)(1 - q_b) - \frac{q_b}{3}(2 - 3q_a) = q_a + \frac{1}{3}q_b
\]

\[
1 - (1 - q_a)(1 - q_b) - \frac{q_b(1 - q_a)^3}{3q_a^2}(3q_a - 1) = \frac{3q_a^4 - 7q_a^3q_b + 12q_a^2q_b - 6q_aq_b + q_b}{3q_a^3}
\]

The right hand of the top equation is the evaluator payoff whenever \( q_a \leq \frac{1}{2} \). Thus, the full information payoff is higher. Similarly, the right hand side of the bottom equation is the evaluator expected payoff when \( q_a > \frac{1}{2} \). Again the full-information expected payoff is higher.

Evaluator payoff comparison with endogenous investment. Here we compare the evaluator’s payoff under fully revealing and strategic grading, when investment is endogenous. The following inequalities define the region of \( (\rho_a, \rho_\beta) \) for which the evaluator payoff is higher with strategic grading, assuming endogenous investment.

\[
\begin{align*}
\frac{(3\rho_a + \rho_\beta)\sqrt{\rho_a \rho_\beta}}{6\rho_a} - (1 - (1 - \frac{\rho^2}{4})(1 - \frac{\rho^2}{4})) & \geq 0 \quad \text{if } \rho_a \rho_\beta \leq 1 \\
\frac{(3\rho_a^2 - 7\rho_a \rho_\beta - 24)\sqrt{\rho_a \rho_\beta} + 24\rho_a \rho_\beta + 8}{6\rho_a^2} - (1 - (1 - \frac{\rho^2}{4})(1 - \frac{\rho^2}{4})) & \geq 0 \quad \text{if } \rho_a \rho_\beta > 1
\end{align*}
\]
This function is continuous. We establish that these inequalities are satisfied along the diagonal $\rho_\alpha = \rho_\beta$, and therefore hold in a region around the diagonal. Let $\rho_\alpha = \rho_\beta = \rho$. These inequalities become:

\[
\begin{align*}
\frac{1}{16} \rho^4 - \frac{1}{2} \rho^2 + \frac{2}{3} \rho & \geq 0 \quad \text{if } \rho \leq 1 \\
\frac{1}{38\rho^2} (2 - \rho)^2 (2 - \rho)^4 + 2 \rho^2 (\rho - \sqrt{3} + 5)(\rho + \sqrt{3} + 5) & \geq 0 \quad \text{if } 1 \leq \rho \leq 2
\end{align*}
\]

The first inequality is satisfied because, $\frac{2}{3} \rho - \frac{1}{2} \rho^2 \geq 0$ for all $0 \leq \rho \leq \frac{4}{3}$. The second inequality is also satisfied because in the region of interest, all terms in the sum are positive. Thus, around the diagonal, the evaluator payoff is higher under strategic grading than under fully-informative grading. Plotting these regions using a software package capable of implicit plots shows that the inequalities hold in an arc-shaped region around the diagonal.
APPENDIX II: INTENDED FOR ONLINE PUBLICATION

B.1 NO GRADES/RAMPANT GRADE INFLATION

If grades or transcripts are banned, then the evaluator assigns the job based solely on the investment level of the school. An identical outcome would follow if for some exogenous reason grades become so inflated that they convey no information about student ability. This game is essentially a full-information all-pay auction with an asymmetric convex cost of bidding, and a bid cap. Each school simultaneously chooses $q_i \in [0, 1]$. The school with the higher value of $q$ receives a payoff of one, but both schools lose their investments, $C_i(q_i) = \frac{q_i^2}{\rho_i}$. We describe the equilibrium of this game in a series of Lemmas.

Lemma B.1 If $\rho_\beta \geq \sqrt{2}$ then $q_\alpha = q_\beta = 1$ is the unique equilibrium.

Proof. Suppose that school $j$ chooses $q_j = 1$. All pure strategies $q_i \in (0, 1)$ lead to payoff zero. Because investment is costly, and given school $j$’s strategy, the best deviation from $q_i = 1$ is therefore $q_i = 0$. Thus, provided $\frac{1}{2} - \frac{1}{\rho_i} \geq 0 \iff \rho_i \geq \sqrt{2}$ $q_i = 1$ is a best response to $q_j = 1$. Because $\rho_\beta \leq \rho_\alpha$, if $\rho_\beta \geq \sqrt{2}$ then $q_\alpha = q_\beta = 1$ is the unique Nash equilibrium.

Lemma B.2 If $1 < \rho_\beta < \sqrt{2}$ then the mixed strategy Nash equilibrium is as follows:

$$q_\alpha = \begin{cases} Q \sim F(x) = \frac{x^2}{2-\rho_\alpha^2} & \text{with probability } \frac{\rho_\alpha}{\rho_\beta} - 1 \\ 1 & \text{with probability } 2(1-\frac{1}{\rho_\beta}) \end{cases}$$

$$q_\beta = \begin{cases} 0 & \text{with probability } 1 - \frac{(\rho_\alpha)^2}{\rho_\beta} \\ Q \sim F(x) = \frac{x^2}{2-\rho_\beta^2} & \text{with probability } (\frac{\rho_\alpha}{\rho_\beta})^2(\frac{1}{\rho_\beta} - 1) \\ 1 & \text{with probability } 2(1-\frac{1}{\rho_\beta})(1-\frac{1}{\rho_\beta}) \end{cases}$$

Proof. Under the parameter range in the proposition, all values we claim are probabilities are in $[0, 1]$ and sum to one. Also, $F(x)$ is increasing on the support of $Q$ which is $[0, \sqrt{2-\rho_\beta^2}]$. To show that the proposed strategies constitute a mixed strategy Nash equilibrium, we verify that each school is indifferent among all pure strategies inside the support and that no pure strategy outside the support delivers a better expected payoff against the mixed strategy of the other player. A derivation of the equilibrium from the indifference conditions, which also shows uniqueness, is available upon request.

Consider school $\beta$’s expected payoff from a pure strategy $p$ in the support of its mixed strategy:

$$u_\beta = \begin{cases} 0 & \text{if } p = 0 \\ \frac{\rho_\beta - 1}{\rho_\beta} F(p) - \frac{p^2}{\rho_\beta} & \text{if } p \in [0, \sqrt{2-\rho_\beta^2}] \\ \frac{2(1-\frac{1}{\rho_\beta})}{2} - \frac{1}{\rho_\beta} & \text{if } p = 1 \end{cases}$$
Substituting and simplifying gives:

\[ u_\beta = \begin{cases} 
0 & \text{if } p = 0 \\
\frac{2}{\rho_\beta} - 1 - \frac{p^2}{\rho_\beta^2} - \frac{p^2}{\rho_\beta^2} = 0 & \text{if } p \in [0, \sqrt{2 - \rho_\beta^2}] \\
1 - \frac{1}{\rho_\beta^2} - \frac{1}{\rho_\beta^2} = 0 & \text{if } p = 1
\end{cases} \]

Thus all pure strategies in the support of school \( \beta \) mixed strategy give the school expected payoff zero. Choosing any pure strategy \( \hat{q} \in (\sqrt{2 - \rho_\beta^2}, 1) \) dominated by choosing \( q = \sqrt{2 - \rho_\beta^2} \), because the probability of winning is the same for both pure strategies, but \( q \) is less costly. Thus all pure strategies in the support of \( \beta \) mixed strategy give the same expected payoff against a mixed strategy, and no strategy outside the support gives \( \beta \) a higher payoff. Thus, \( \beta \) mixed strategy is a best response to \( \alpha \)’s mixed strategy.

Consider school \( \alpha \)’s expected payoff from a pure strategy \( p \) in the support of its mixed strategy:

\[ u_\alpha = \begin{cases} 
1 - \left(\frac{p_\alpha}{\rho_\alpha}\right)^2 + \left(\frac{p_\alpha}{\rho_\alpha}\right)^2 \left(\frac{2}{\rho_\alpha} - 1\right) F(p) - \frac{p^2}{\rho_\alpha^2} & \text{if } p \in [0, \sqrt{2 - \rho_\alpha^2}] \\
1 - \frac{1}{\rho_\alpha^2} - \frac{1}{\rho_\alpha^2} = 1 - \left(\frac{p_\alpha}{\rho_\alpha}\right)^2 & \text{if } p = 1
\end{cases} \]

Substituting and simplifying gives:

\[ u_\alpha = \begin{cases} 
1 - \left(\frac{p_\alpha}{\rho_\alpha}\right)^2 + \left(\frac{p_\alpha}{\rho_\alpha}\right)^2 \left(\frac{2}{\rho_\alpha} - 1\right) \frac{2}{2 - \rho_\alpha^2} - \frac{p^2}{\rho_\alpha^2} = 1 - \left(\frac{p_\alpha}{\rho_\alpha}\right)^2 & \text{if } p \in [0, \sqrt{2 - \rho_\beta^2}] \\
1 - \frac{1}{\rho_\alpha^2} = 1 - \left(\frac{p_\alpha}{\rho_\alpha}\right)^2 & \text{if } p = 1
\end{cases} \]

Thus all pure strategies in the support of school \( \alpha \) mixed strategy give the school expected payoff \( 1 - \left(\frac{p_\alpha}{\rho_\alpha}\right)^2 \), equal to the probability that school \( \beta \) plays \( q = 0 \). Choosing any pure strategy \( \hat{q} \in (\sqrt{2 - \rho_\beta^2}, 1) \) dominated by choosing \( q = \sqrt{2 - \rho_\beta^2} \), because the probability of winning is the same for both pure strategies, but \( q \) is less costly. Thus all pure strategies in the support of \( \alpha \) mixed strategy give the same expected payoff against \( \beta \) mixed strategy, and no strategy outside the support gives \( \alpha \) a higher payoff. Thus, \( \alpha \) mixed strategy is a best response to \( \beta \)’s mixed strategy. ■

**Lemma B.3** If \( \rho_\beta \leq 1 \) then the mixed strategy Nash equilibrium is as follows:

\[ q_\alpha = Q \sim F(x) = \frac{x^2}{\rho_\beta^2} \]

\[ q_\beta = \begin{cases} 
0 & \text{with probability } 1 - \left(\frac{p_\alpha}{\rho_\alpha}\right)^2 \\
Q \sim F(x) = \frac{x^2}{\rho_\beta^2} & \text{with probability } \left(\frac{p_\alpha}{\rho_\alpha}\right)^2
\end{cases} \]

**Proof.** To show that the proposed strategies constitute a mixed strategy Nash equilibrium, we verify that each school is indifferent among all pure strategies inside the support and that no pure strategy outside the support delivers a better expected payoff against the mixed strategy of the other player. A derivation of the equilibrium from the indifference conditions, which also shows uniqueness, is available upon request.

Consider school \( \beta \)’s expected payoff from a pure strategy \( p \) in the support of its mixed strategy:

\[ u_\beta = \begin{cases} 
0 & \text{if } p = 0 \\
F(p) - \frac{p^2}{\rho_\beta^2} & \text{if } p \in [0, \rho_\beta]
\end{cases} \]
Substituting and simplifying gives that in both cases \( u_\beta = 0 \). Thus all pure strategies in the support of school \( \beta \) mixed strategy give the school expected payoff zero. Choosing any pure strategy \( \hat{q} \in (\rho_\beta, 1) \) is dominated by choosing \( q = \rho_\beta \), because the probability of winning is the same for both pure strategies, but \( q \) is less costly. Thus all pure strategies in the support of \( \beta \) mixed strategy give the same expected payoff against \( \alpha \) mixed strategy, and no strategy outside the support gives \( \beta \) a higher payoff. Thus, \( \beta \) mixed strategy is a best response to \( \alpha \)'s mixed strategy.

Consider school \( \alpha \)'s expected payoff from a pure strategy \( p \) in the support of its mixed strategy:

\[
    u_\alpha = 1 - \left( \frac{\rho_\alpha}{\rho_\beta} \right)^2 + \left( \frac{\rho_\alpha}{\rho_\beta} \right)^2 F(p) - \frac{\rho_\alpha^2}{\rho_\beta^2} \quad \text{if} \quad p \in [0, \rho_\beta]
\]

Simplifying gives that all pure strategies in the support of school \( \alpha \) mixed strategy give the school expected payoff \( 1 - \left( \frac{\rho_\beta}{\rho_\alpha} \right)^2 \), equal to the probability that school \( \beta \) plays \( q = 0 \). Choosing any pure strategy \( \hat{q} \in (\rho_\beta, 1) \) is dominated by choosing \( q = \rho_\beta \), because the probability of winning is the same for both pure strategies, but \( q \) is less costly. Thus all pure strategies in the support of \( \alpha \) mixed strategy give the same expected payoff against \( \beta \) mixed strategy, and no strategy outside the support gives \( \alpha \) a higher payoff. Thus, \( \alpha \) mixed strategy is a best response to \( \beta \)'s mixed strategy.

**Banning Grades.** Here we show that banning grades outright may be preferred to either strategic grading or fully-revealing grading. In order to make this calculation, we need to know the evaluator payoff when grades are banned. In the interest of space, we make the calculation along the diagonal, \( \rho_\alpha = \rho_\beta = \rho \). If the results hold on the diagonal, by continuity, they also hold in a region off-diagonal.

**Case I:** \( 1 \leq \rho \leq \sqrt{2} \) Observe that

\[
    E[Q^{(2)}] = 2 \int_0^{\sqrt{2-\rho^2}} \left( \frac{2x}{2-\rho^2} \right) \left( \frac{x^2}{2-\rho^2} \right) (x) dx = \frac{4}{5} \sqrt{2-\rho^2}
\]

Where \( Q^{(2)} \) represents the maximum of two draws of \( Q \). The evaluator expected payoff is therefore

\[
    1 - \left( \frac{2}{\rho^2} - 1 \right)^2 + \left( \frac{2}{\rho^2} - 1 \right)^2 E[Q^{(2)}] = \frac{4}{5\rho^4} \left( (2 - \rho^2)^{\frac{3}{2}} - 5(1 - \rho^2) \right)
\]

**Case II:** \( \rho \leq 1 \) Observe that the evaluator expected payoff is given by:

\[
    E[Q^{(2)}] = 2 \int_0^\rho \left( \frac{2x}{\rho^2} \right) \left( \frac{x^2}{\rho^2} \right) (x) dx = \frac{4}{5} \rho
\]

A simple plot reveals that these payoffs dominate both fully-revealing and strategic grading along the diagonal.

**B.2 UNIQUENESS OF STRATEGIC GRADING EQUILIBRIUM (LEMMA 4.2)**

In this section we derive the equilibrium of the stage 2 game with general grading policies from first principles and demonstrate that the equilibrium is unique.
A strategy for player $j$, is a random variable $\Gamma_j$ with support contained in the unit interval, and expectation $q_j$. Furthermore, (as discussed above) because the underlying signal structure is valid, $\Gamma_j$ has a finite number of mass points $m_{jk}^i$, contained in set $M_j$. Let $\mu_k^j = Pr(\Gamma_j = m_{jk}^i)$. Ignoring mass points, random variable $\Gamma_j$ has support on a (closed) interval $I_j$. Denote support$(\Gamma_j) = I_j \cup M_j$ as $S_j$.

Let $W_j(x)$ represent the probability that graduate $i$ is selected when $i$ posterior belief realization is equal to $x$.

$$W_j(x) = Pr(\Gamma_j < x) + \frac{1}{2} Pr(\Gamma_j = x)$$

Thus, for any point inside $x \in S_j$ this function is given by the following expression:

$$W_j(x) = \begin{cases} 
  P_j(x) & \text{if } x \in I_j \cap M_j^C \\
  P_j(x) - \frac{1}{2} \mu_k^j & \text{if } x = m_{jk}^i
\end{cases}$$

Here we use $X^C$ to represent the complement of set $X$. Also note that function $W_j(x)$ maintains a constant value in any interval that does not intersect $S_j$. This is because $P_j(x)$ is neither increases nor decreases outside of set $S_j$.

As is the case for $\Gamma_j$ the support of a strategy for player $i$, is $S_i = I_i \cup M_i$, where $I_i$ represents an interval and $M_i$ represents a finite set of mass points; $m_{ik}^i$ represents mass point $k$ and $\mu_k^i$ represents the mass on point $m_{ik}^i$. The best response of player $i$ to $\Gamma_j$ is a choice of random variable $\Gamma_i$ with generalized density $p_i(x)$ to solve the following maximization:

$$\max \int_0^1 p_i(x) W_j(x) dx$$

subject to

$$\int_0^1 p_i(x) dx = q_i$$

Consider the Lagrangian for this problem:

$$L = \int_0^1 p_i(x)(W_j(x) - \lambda_i(x - \gamma_i)) dx$$

Standard maximization principles require that the value of the integrand of the Lagrangian is the same at any value of $x$ inside the support of $\Gamma_i$ and no value outside of the support of $\Gamma_i$ gives a higher value. Therefore:

$$L_i(x) = W_j(x) - \lambda_i(x - q_i)$$

(5)

$$x \in S_i \Rightarrow L_i(x) = v_i$$

(6)

$$x \notin S_i \Rightarrow L_i(x) \leq v_i$$

Equation 6 implies several properties of best-responses.

**Properties of best responses**

1. If $\Gamma_i$ is a best response to $\Gamma_j$, its interval support $I_i$ is a weak subset of the interval support of $\Gamma_j$:
\( I_i \subseteq I_j \)

Suppose \( I_i \cap I_j^C \) is non-empty. If so, it contains an interval. Outside \( I_j \), only support of \( \Gamma_j \) is a finite set of mass points. Hence, exists a subinterval in \( I_i \cap I_j^C \) that does not intersect \( S_j \). On this interval, however, \( W_j(x) \) is constant, while \( \lambda_i(x - q_i) \) is increasing, and \( L_i(.) \) is non-constant. Contradicts equation (6).

2. For any best response \( \Gamma_i \), if \( \Gamma_j \) has a mass point on \( m_j \in M_j \), then there exists \( \epsilon \) such that \( (m_j - \epsilon, m_j) \) does not intersect \( I_i \).

Because \( W_j(x) \) jumps up at \( m_j \) but \( \lambda_i x \) does not, \( L_i(x) \) jumps up at \( m_j \). Hence, any value of \( x \geq m_j \) can not give the same value of \( L(.) \) as a value of \( x \in (m_j - \epsilon, m_j) \).

3. If \( \Gamma_j \) does not have a mass point on 1 and the right endpoint of \( I_j \) is strictly less than 1, then player \( i \) best response \( \Gamma_i \) does not have a mass point on 1.

If \( \Gamma_j \) does not have a mass point on 1 and the right endpoint of \( I_j \) is strictly less than 1, then for sufficiently small \( \epsilon \), \( L_i(1 - \epsilon) > L_i(1) \).

These properties have significant implications for the structure of possible equilibria.

**Properties of Equilibrium**

1. In equilibrium no mass point inside \([0, 1]\) can be common to both \( \Gamma_i \) and \( \Gamma_j \). Suppose a mass point exists at \( m \), and \( Pr(\Gamma_j = m) = \mu \). \( W_j(.) \), the probability of winning for player \( i \), jumps up at \( m \) by \( \mu/2 \). The second component of the Lagrangian, \( \lambda_i(x - q_i) \), is continuous. Hence, \( L_i(.) \) jumps up at \( m \) by \( \mu/2 \) i.e. for any \( \epsilon > 0 \), \( L_i(m + \epsilon) > L_i(m) \). If exists \( \epsilon \) for which \( m + \epsilon < 1 \), then this contradicts condition (6). Hence, only possible common mass point is \( m = 1 \).

2. In equilibrium \( I_i = I_j \).

Direct consequence of point 1. in properties of best responses.

3. In equilibrium, the smallest element of the (identical) interval support \( I \), is 0.

In equilibrium, both random variables are supported on the same interval \( I \). Suppose the smallest element of \( I \), denoted \( x \) is strictly above 0. Because no common mass point exists in \([0, 1]\) at most one of \( \Gamma_i \) and \( \Gamma_j \) can have a mass point on \( x \). If exactly one has a mass point on \( x \), let \( \Gamma_i \) be the random variable with no mass point on \( x \). Besides interval \( I \), \( \Gamma_i \) is supported on a set of mass points. This implies that for sufficiently small \( \epsilon \), no mass point exists between \( x - \epsilon \) and \( x \). For \( \Gamma_i \), no mass point exists on \( x \), and, because \( x \) is the smallest element of \( I \), \( F_j(x) = 0 \). Hence \( W_j(x) = W_j(x - \epsilon) \). However, the second component of the Lagrangian is decreasing. Thus \( L(x - \epsilon) > L(x) \). This contradicts condition 6.

4. In equilibrium, no mass point exists in \((0, 1)\) for either player.

In equilibrium each player’s strategy has the same interval support \( I = [0, r] \). By point 2. of properties of best responses, no mass point can exist in interval \((0, r)\). Otherwise, a gap must exist in the interior of \( I \). However, if a player has mass point above \( r \), then it must be shared with the other player. If it is not shared with the other player, then a point just below generates the same winning probability, but lower \( \lambda_i(x - q_i) \) and hence a greater value of \( L(.) \). However, according to property of equilibrium 1, only possible common mass point is 1.
5. In equilibrium, if $\Gamma_i$ has a mass point on 1, then $\Gamma_j$ also has a mass point on 1.

If $i$ has a mass point on 1, then there exists an interval $(1 - \epsilon, 1)$ outside of the support of $\Gamma_j$. If no mass point on 1 is part of $j$'s strategy, then, because the probability of $i$ winning is constant outside of $S_j$, then there exists $\epsilon$ for which $L_i(1 - \epsilon) > L_i(1)$, contradicting condition (6).

These conditions partially characterize equilibrium strategies: Conditions 1.-4. characterize the structure of equilibrium strategies. An equilibrium strategy for player $k \in \{a, b\}$ must have the following structure:

$$\Gamma_k = \begin{cases} 
\Phi_k & \text{with probability } m_k \\
1 & \text{with probability } n_k
\end{cases}$$

where $m_k + n_k = 1$ and $\Phi_k$ is a random variable with support over an interval $I = [0, r]$ and no mass points, except possibly at 0. Note, however, that at most one equilibrium strategy can have a mass point at zero. The CDF of $\Phi_k$ is given by $F_k(x)$, where $F_k(x)$ is continuous (and differentiable) and $F_k(0) \geq 0$ and $F_k(r) = 1$. In this case the win-probability for player $i$ has the following structure:

$$W_j(x) = \begin{cases} 
m_k F_j(x) & \text{if } x \in [0, r] \\
1 - n_j & \text{if } x = 1
\end{cases}$$

The above structure allows us to simplify condition (6). For $i, j \in \{a, b\}$ and $i \neq j$, condition (6) reduces to the following:

$$(7) \quad m_j F_j(x) - \lambda_i(x - q_i) = v_i \text{ for all } x \in [0, r]$$

$$n_i > 0 \implies 1 - \frac{n_j}{2} - \lambda_i(1 - q_i) = v_i$$

The first part of this condition therefore implies that:

$$F_j(x) = \frac{v_i - \lambda_i q_i}{m_j} + \frac{\lambda_i x}{m_j}$$

$$F_i(x) = \frac{v_i - \lambda_j q_j}{m_i} + \frac{\lambda_j x}{m_i}$$

Thus, random variable $\Phi_k$ must be uniformly distributed with a possible mass point on 0. However, because it cannot be that both $i, j$ have a mass point on 0 (No common mass points in $[0, 1]$), at most one player has a mass point on 0. Let this be player $j$.

$$F_i(0) = 0 \text{ and } F_j(0) \geq 0 \iff \frac{v_j - \lambda_j q_j}{m_i} = 0 \text{ and } \frac{v_i - \lambda_i q_i}{m_j} \geq 0.$$ 

Neither player can have a mass point at $r$. Therefore,

$$F_i(r) = F_j(r) = 1 \iff \frac{v_j - \lambda_j q_j}{m_i} + \frac{\lambda_j}{m_i} r = 1 \text{ and } \frac{v_i - \lambda_i q_i}{m_j} + \frac{\lambda_i}{m_j} r = 1$$
In addition each strategy must satisfy the appropriate mean constraint. Given the above conditions,

\[ E[\Phi_i] = \frac{\lambda_ir^2}{2m_i} \quad \text{and} \quad E[\Phi_j] = \frac{\lambda_ir^2}{2m_j} \]

Therefore the mean constraints are:

\[ m_iE[\Phi_i] + n_i = q_i \leftrightarrow \lambda_i(\frac{r^2}{2}) + n_i = q_i \]
\[ m_jE[\Phi_j] + n_j = q_j \leftrightarrow \lambda_i(\frac{r^2}{2}) + n_j = q_j \]

Thus every equilibrium must satisfy the following conditions (SC) (collected from above):

\[ \frac{v_j - \lambda_jq_j}{m_i} = 0 \quad \text{and} \quad \frac{v_i - \lambda_iq_i}{m_j} \geq 0 \]
\[ \frac{v_j - \lambda_jq_j}{m_i} + \frac{\lambda_j}{m_i}r = 1 \quad \text{and} \quad \frac{v_i - \lambda_iq_i}{m_j} + \frac{\lambda_i}{m_j}r = 1 \]
\[ \lambda_i(\frac{r^2}{2}) + n_i = q_i \quad \text{and} \quad \lambda_i(\frac{r^2}{2}) + n_j = q_j \]
\[ m_i+n_i = 1 \quad \text{and} \quad m_j+n_j = 1 \]

Next, observe that the equilibrium properties imply that only two equilibrium structures are possible. In
the first, \( n_i = n_j = 0 \), and in the other \( n_i > 0 \) and \( n_j > 0 \).

**Case I:** \( n_i = n_j = 0 \Rightarrow m_i = m_j = 1 \). In this case, system (SC) reduces to the following:

\[ v_j = \lambda_jq_j \quad \text{and} \quad v_i \geq \lambda_iq_i \]
\[ \lambda_jr = 1 \quad \text{and} \quad v_i - \lambda_iq_i + \lambda_ir = 1 \]
\[ \lambda_j(\frac{r^2}{2}) + q_i \quad \text{and} \quad \lambda_i(\frac{r^2}{2}) = q_j \]

The unique solution of this system is

\[ r = 2q_i \quad \text{and} \quad \lambda_i = \frac{q_j}{2q_i^2} \quad \text{and} \quad \lambda_j = \frac{1}{2q_i} \quad \text{and} \quad v_i = \frac{q_j-2q_i}{2q_i} \quad \text{and} \quad v_j = \frac{q_j}{2q_i} \]

Finally, \( v_i - \lambda_iq_i = 1 - q_j/q_i \). In order for the required inequality to hold, it must be that \( q_j \leq q_i \); hence,
school A plays the role of school i. This is exactly the equilibrium described in the first part of Lemma 4.2,
and this equilibrium is therefore unique.

**Case II:** \( n_i > 0, n_j > 0 \). In addition to conditions (SC), we also obtain the following indifference
conditions:

\[ 1 - \frac{n_j}{2} - \lambda_i(1 - q_i) = v_i \quad \text{and} \quad 1 - \frac{n_i}{2} - \lambda_j(1 - q_j) = v_j \]

The solution of the system of equations is given by:

\[ r = 2(1 - q_i) \quad \text{and} \quad \lambda_i = \frac{1}{2q_j^2} \quad \text{and} \quad \lambda_j = \frac{1}{2q_i} \quad \text{and} \quad m_i = \frac{1}{q_i} - 1 \quad \text{and} \quad m_j = \frac{q_j+q_j^2-2q_iq_j}{q_i^2} \]
\[ n_i = 2 - \frac{1}{q_i} \quad \text{and} \quad n_j = \frac{q_j}{q_i}(2 - \frac{1}{q_i}) \]

Finally, \( v_i \geq \lambda_i q_i \Rightarrow q_i \geq q_j \). Hence, school A plays the role of school i. This is exactly the equilibrium described in the second part of the Lemma 4.2, and this equilibrium is therefore unique.